Lecture Notes

on

Power System Stability

Mrinal K Pal
CONTENTS

Preface ix

Chapter 1  Fundamentals of Power Flow and Power Limits  1-1
   Representation of Transmission Lines  1-1
   Equations of a transmission line  1-1
      Equivalent circuit of a transmission line  1-3
      \textup{Y} - \textup{V}  transformation  1-5
   Per Unit System  1-6
   Base quantities  1-6
      Change of bases  1-7
   Per unit and percent admittance  1-8
   Power Limits  1-9
   Circle Diagram  1-11
   Basic Power Flow Calculation  1-12
      Direct solution  1-12
      Iterative procedure (Newton’s method)  1-13
      Selection of starting values  1-14
   Steady State Stability  1-14
      Steady state stability limit  1-15
      Steady state stability criterion  1-17
      Steady state stability of a two-machine system  1-17
   General Network Equations  1-20
      Nodal formulation in terms of bus admittance matrix  1-20
      General equations for real and reactive power  1-21
      Elimination of network nodes by matrix method  1-22
   Application of the Steady-State Stability Criterion to an \( n \) Machine System  1-23
   Network Calculations Using Bus Impedance Matrix  1-24
      Adding a new bus to the network  1-25
      Adding a new line between two existing buses  1-26
      Adding a new line between an existing bus and neutral (reference bus)  1-27
   Calculation of Short-Circuit Current  1-28
   Representation of Tap-changing Transformers and Phase-shifting Transformers  1-29
      Effect of phase shift on power flow  1-30
   References  1-32

Chapter 2  Stability Basics  2-1
   Stability of Dynamic Systems  2-1
      Stability definitions  2-1
      Power system stability definitions  2-2
      Comments  2-2
      Small-disturbance stability  2-2
      Large-disturbance stability  2-3
   Linear and Non-Linear Systems  2-3
      Linearization of non-linear systems  2-3
Solution of linear differential equations with constant coefficients  2-5
Evaluation of the arbitrary constants  2-6
Criterion for stability of a linear system  2-6
Concepts of damping ratio and natural frequency  2-7
The State Variables of a Dynamic Systems  2-9
The state vector differential equation  2-11
Solution of the equation \( \mathbf{x}' = \mathbf{A}\mathbf{x} \)  2-13
Liapunov’s Direct Method of Stability Analysis  2-17
Stability and instability theorems  2-17
Illustration of positive definite functions and their closedness  2-18
Stability determined by linear approximation  2-19
Extent of asymptotic stability  2-20
Construction of Liapunov functions  2-20
Comments of Certain Terms in Common Use  2-21
References  2-23

Chapter 3  Synchronous Machine Stability Basics

The Swing Equation  3-1
Selection of units  3-3
Mechanical Torque  3-4
Unregulated machine  3-4
Regulated machine  3-5
Small Disturbance Performance of Unregulated Synchronous Machine System  3-6
Single machine connected to infinite bus  3-6
Modes of Oscillation of an Unregulated Multi-Machine System  3-8
Division of Suddenly Applied Load Among Generators in the System  3-15
Transient Stability by Equal Area Criterion  3-19
Effect of damping  3-22
Equal area criterion for a two-machine system  3-22
Transient Stability by Liapunov’s Method  3-23
References  3-26

Chapter 4  Numerical Solution of the Transient Stability Problem

Numerical Solution of Differential Equations in Power System Stability Studies  4-1
Euler’s method  4-2
The modified Euler’s method  4-3
Runge-Kutta method  4-3
Predictor-Corrector method  4-4
Implicit multi-step integration  4-5
Solution of Multi-Machine Transient Stability Problem  4-5
Using Classical Machine Model  4-5
Single-machine-infinite bus system  4-6
Multi-machine system  4-7
Handling of constant current load  4-10
Solution of Faulted Networks  4-14
Analysis of Unsymmetrical Faults  4-16
Single line-to-ground fault  4-17
CONTENTS

Line-to-line fault 4-17
Double line-to-ground fault 4-17
One open conductor 4-18
References 4-19

Chapter 5  Synchronous Machines
Voltage Relations 5-2
Flux-Linkage Relations 5-3
Inductance Relations 5-4
Armature self-inductances 5-4
Armature mutual inductances 5-5
Rotor self-inductances 5-5
Rotor mutual inductances 5-5
Mutual inductances between stator and rotor circuits 5-5
Transformation of Equations 5-6
Per Unit System 5-8
Choice of rotor base currents 5-14
Per unit voltage equations 5-15
Per unit flux-linkage equations 5-16
Power and Torque 5-16
Steady State Operation 5-17
Locating the $d$ and $q$ axes 5-19
Open-circuit operation 5-19
Reactive Power 5-20
Subtransient and Transient Reactances and Time Constants 5-20
Synchronous Machine Models for Stability Studies 5-24
Model 1 -- One damper winding on each axis 5-26
Model 2 -- One equivalent damper winding on the $q$ axis only 5-31
Model 3 -- no damper winding 5-32
Saturation 5-32
Saturation in cylindrical rotor machines 5-33
Saturation in salient pole machines 5-34
Inclusion of the effect of saturation in machine modeling 5-35
Network Model in $DQ$ Reference Frame 5-37
Transformation from machine to network reference frame 5-38
Stability Computation in Multi-Machine Systems 5-39
An iterative method for handling saliency 5-42
Generator Capability Curve 5-43
Stability Limit at Constant Field Voltage 5-44
Reactive power limit at $P = 0$ 5-48
Self Excitation 5-48
References 5-52

Chapter 6  Effect of Excitation Control on Stability
Effect of Excitation on Generator Power Limit 6-1
Effect of Excitation Control on Small-Disturbance Performance of Synchronous Machines 6-3
CONTENTS

Synchronizing power coefficient with constant flux linkage 6-7
Steady-state synchronizing power coefficient 6-7
Evaluation of damping torque 6-9
   No voltage regulator action 6-9
   Including voltage regulator action 6-11
Effect of Damper Windings 6-15
   One damper winding on the $q$ axis, no damper winding on the $d$ axis 6-15
   Evaluation of damping torque 6-17
      No voltage regulator action 6-17
      Effect of voltage regulator 6-18
   One damper winding on the $d$ axis, two damper windings on the $q$ axis 6-19
   Evaluation of damping torque 6-21
      No voltage regulator action 6-21
      Effect of voltage regulator 6-22
Supplementary Stabilizing Signals 6-22
References 6-25

Chapter 7  Representation of Loads in Stability Studies

Load Characteristics 7-1
Load Modeling Concepts 7-1
Static Model for Stability Studies 7-2
   Discussion 7-3
Motor Load 7-5
Induction Motor 7-5
Induction Motor Representation in Stability Studies 7-12
   Initial condition calculation 7-15
Generic Load Model 7-16
References 7-17

Chapter 8  Small Disturbance Stability

State Space Representation of Power System 8-3
   Single machine-infinite bus system 8-3
   State model of the multi-machine system 8-5
An Efficient Method of Deriving the State Model 8-8
   State model using flux linkages as state variables 8-8
   State model for machine model 2 8-13
   Mixed machine models 8-14
   State model for systems containing induction motor loads 8-16
   Singularity of the coefficient matrix 8-18
Procedure for Small-Disturbance Stability Studies 8-18
Eigen-sensitivity Analysis 8-19
System Performance 8-20
References 8-23

Chapter 9  Turbine-Generator Shaft Torsionals

A Basic Analysis of the Shaft Torque Problem 9-2
Modes and Mode Shapes in a Turbine-Generator Shaft Torsional System 9-7
CONTENTS

<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitor switching</td>
<td>10-53</td>
</tr>
<tr>
<td>Extending Voltage Stability Limit by SVC</td>
<td>10-55</td>
</tr>
<tr>
<td>LTC Operation and Voltage Stability</td>
<td>10-58</td>
</tr>
<tr>
<td>Voltage Stability Studies in Actual Power Systems</td>
<td>10-62</td>
</tr>
<tr>
<td>A Typical Voltage Collapse Scenario</td>
<td>10-63</td>
</tr>
<tr>
<td>References</td>
<td>10-64</td>
</tr>
<tr>
<td>Appendix A  Review of Matrices</td>
<td></td>
</tr>
<tr>
<td>Appendix B  Review of Selected Topics from Control Theory</td>
<td></td>
</tr>
<tr>
<td>Appendix C  Block Diagram State Model</td>
<td></td>
</tr>
</tbody>
</table>
PREFACE

The materials presented in this work are based on the notes used in courses taught sporadically to graduate students and practicing engineers over a period of thirty-five years. The objective here is to present the fundamental materials and concepts in power system stability in a tutorial form. The materials covered were dictated by the need of the course attendees. This is certainly not an exhaustive treatment of all aspects of the subject found in the literature. Some of the topics usually found in text books on power system stability are not covered. For example, detailed descriptions and workings of excitation and governor control systems are not included. However, the effect of excitation control on stability has been covered in some detail.

Chapter 1 includes some basic material considered important for developing a proper understanding of the subject. In the days of the extensive use of commercially available large-scale stability programs, the topics discussed in this chapter might be considered superfluous. However, a good understanding of the basics not only helps in developing a clearer concept of the subject, but also allows a quick estimate of what is to be expected from a computer run, thus enabling a check on the proper working of the program as well as on the correctness and validity of the data entered.

It will be noted that no mention of “FACTS” has been made anywhere in the book. This is because the concepts and underlying principles involved in the application of devices e.g., series and shunt capacitors, phase-shifting transformers, etc., are not new. It is the use of power electronics, that provides fast and smooth continuous control, that is new. In the future newer technologies will appear that will provide even better control. The basic principles and concepts involved in applying these technologies, and the methods of analyses, are discussed throughout the book. Anyway, it is not clear why a terminology like “FACTS” was adopted in the first place, when these are really applications of power electronics.

The preparation of the book, especially typing the equations, was tedious and typos are inevitable. I would be grateful if the readers would report them, and any other problem they might encounter, to me for future correction. I can be reached at (732) 494 3716 and at m.k.pal@ieee.org.

I would like to thank Mr. Rondeep Pal for converting the document into the PDF format, and providing help in many other ways.

Mrinal K Pal

Edison, New Jersey
June, 2007
CHAPTER 1
FUNDAMENTALS OF POWER FLOW AND POWER LIMITS

Representation of Transmission Lines
A transmission line has four parameters -- series resistance and inductance, and shunt conductance and capacitance. These are uniformly distributed along the line. In the derivation of transmission line equations the following assumptions are usually made:

The line is transposed (or symmetrical), and the three phases are balanced. In practice this is not entirely true but the unbalance and departure from symmetry are small. The line can therefore be analyzed on a per-phase basis.

The line parameters, per unit length, are constant, i.e., they are independent of position, frequency, current, and voltage. This assumption, although approximate, is permissible for most power system studies.

Equations of a transmission line
The voltage and current on a transmission line depend, in general, upon both time and position. Consequently, a general mathematical description of the line involves partial differential equations.

A transmission line section is shown schematically in Figure 1.1. The voltage and current conditions at a small section of length $dx$ at a distance $x$ from the receiving end of the line is as shown in the figure. The voltage and current, denoted by $v$ and $i$ respectively, represent instantaneous quantities. Denoting the series resistance and inductance and the shunt conductance and capacitance per unit length of the line by $r$, $l$, $g$ and $c$, respectively, the series voltage for the section $dx$, is

$$
\frac{\partial v}{\partial x} dx = r dx \frac{\partial i}{\partial t} + l dx \frac{\partial i}{\partial t}
$$

(1.1)

and the shunt current is

$$
\frac{\partial i}{\partial x} dx = g dx v + c dx \frac{\partial v}{\partial t}
$$

(1.2)

Fig. 1.1 Schematic representation of a transmission line.
Equations (1.1) and (1.2) apply to both steady state and transient conditions. For power flow and stability analysis purposes we need to consider only the steady state phenomena. The transmission line transients are fast acting and have negligible impact on system stability except in special situations. The steady state sinusoidal time variation of voltage and current can be represented by

\[ v(x,t) = V(x)e^{j\omega t}, \quad i(x,t) = I(x)e^{j\omega t} \]

Therefore, in the steady state, equations (1.1) and (1.2) reduce to

\[ \frac{dV(x)}{dx} = (r + j\omega l)I(x) = zI(x) \tag{1.3} \]

\[ \frac{dI(x)}{dx} = (g + j\omega c)V(x) = yV(x) \tag{1.4} \]

where \( z \) and \( y \) are the series impedance and shunt admittance per unit length of the line, respectively.

Equations (1.3) and (1.4) can be combined to obtain equations in one unknown, yielding the following equations. In these equations \( V \) and \( I \) represent rms values.

\[ \frac{d^2V}{dx^2} = yzV \tag{1.5} \]

\[ \frac{d^2I}{dx^2} = yzI \tag{1.6} \]

The solutions of equations (1.5) and (1.6) are

\[ V = \frac{V_R + I_R Z_c}{2} e^{j\gamma x} + \frac{V_R - I_R Z_c}{2} e^{-j\gamma x} \tag{1.7} \]

\[ I = \frac{V_R |Z_c| + I_R Z_c}{2} e^{j\gamma x} - \frac{V_R |Z_c| - I_R Z_c}{2} e^{-j\gamma x} \tag{1.8} \]

where

\[ Z_c = \sqrt{\frac{z}{y}} \] is the characteristic impedance

\[ \gamma = \sqrt{yz} \] is the propagation constant

Equations (1.7) and (1.8) give the rms values of \( V \) and \( I \), and their phase angles at any distance \( x \) from the receiving end.

Both \( Z_c \) and \( \gamma \) are complex quantities. \( \gamma \) can be expressed as \( \gamma = \alpha + j\beta \). \( \alpha \) is called the attenuation constant and \( \beta \) the phase constant.

The two terms in equation (1.7) ((1.8)) are called incident voltage (current) and reflected voltage (current), respectively. If a line is terminated in its characteristic impedance \( Z_c \), \( V_R = I_R Z_c \) and there is no reflected voltage or current, as may be seen from equations (1.7) and (1.8). If a line is lossless, its series resistance and shunt conductance are zero and the characteristic impedance
reduces to \( \sqrt{L/C} \), a pure resistance. Under these conditions, the characteristic impedance is often referred to as surge impedance.

The wavelength \( \lambda \) is the distance along a line between two points of a wave which differ in phase of \( 2\pi \) radians. If \( \beta \) is the phase shift in radians per mile, the wavelength in miles is

\[
\lambda = \frac{2\pi}{\beta} \approx \frac{1}{f\sqrt{LC}} \quad (1.9)
\]

where \( L \) and \( C \) are the series inductance and shunt capacitance per mile, respectively.

At a frequency of 60 Hz, the wavelength is approximately 3,000 miles. The velocity of propagation of a wave in miles per second is

\[
v = f\lambda \approx \frac{1}{\sqrt{LC}} \quad (1.10)
\]

Equations (1.7) and (1.8) can be rearranged using hyperbolic functions

\[
sinh = \frac{e^\theta - e^{-\theta}}{2}, \quad \cosh = \frac{e^\theta + e^{-\theta}}{2}
\]

to obtain

\[
V = V_R \cosh \gamma l + I_R Z_c \sinh \gamma l
\]

and

\[
I = I_R \cosh \gamma l + \frac{V_R}{Z_c} \sinh \gamma l
\]

Letting \( x = l \), where \( l \) is the length of the line,

\[
V_s = V_R \cosh \gamma l + I_R Z_c \sinh \gamma l \quad (1.11)
\]

\[
I_s = I_R \cosh \gamma l + \frac{V_R}{Z_c} \sinh \gamma l \quad (1.12)
\]

Equations (1.11) and (1.12) can be solved for \( V_R \) and \( I_R \) to obtain

\[
V_R = V_s \cosh \gamma l - I_s Z_c \sinh \gamma l \quad (1.13)
\]

\[
I_R = I_s \cosh \gamma l + \frac{V_s}{Z_c} \sinh \gamma l \quad (1.14)
\]

Equations (1.11) through (1.14) are the fundamental equations of a transmission line. For balanced three-phase lines, the current is the line current, and the voltage is the line-to-neutral voltage, i.e., the line voltage divided by \( \sqrt{3} \).

**Equivalent circuit of a transmission line**

A transmission line can be represented accurately by a lumped parameter equivalent circuit (either a \( \pi \) or T circuit), insofar as the conditions at the ends of the line are concerned. An equivalent \( \pi \) circuit is shown in Figure 1.2, where the equivalent series impedance and shunt admittance are also shown.
Fig. 1.2 Equivalent π circuit of a transmission line.

Note that $Z = zl$ and $Y = yl$ in Figure 1.2 represent the total series impedance and shunt admittance of the line, respectively. Also note that $\gamma = \sqrt{ZY}$.

Problems

1. Derive the expressions for $Z_\pi$ and $Y_\pi/2$ as shown in Fig.1.2.

2. Derive the equivalent T circuit.

\[
\frac{\sinh \gamma l}{\gamma l} \quad \text{and} \quad \frac{\tanh(\gamma l/2)}{\gamma l/2}
\]

are the factors by which the total line series impedance and shunt admittance are to be multiplied in order to obtain the series impedance and shunt admittances of the equivalent π circuit. The correction factors approach unity as $\gamma l$ (or $ZY$) approaches zero, i.e., as the line becomes electrically shorter.

A π circuit in which the series arm has the impedance $Z$ and each of the shunt arms has the admittance $Y/2$, obtained by setting the correction factors equal to unity, is called a nominal π circuit. A T circuit with two series arms each of impedance $Z/2$ and one shunt arm of admittance $Y$ is called a nominal T circuit. Nominal π and nominal T are approximations to the equivalent π and equivalent T, respectively. The approximations are valid for lines less than 100 miles long. Longer lines may be broken into two or more segments and each segment may be represented by a nominal π or T. A nominal π is more convenient for computational purposes and is therefore more widely used.

Problem

Derive the sending-end voltage ($V_S$) and current ($I_S$) in terms of the receiving-end voltage ($V_R$) and current ($I_R$), and vice-versa, for both the nominal π and the nominal T circuits.

For short transmission lines (less than 50 miles long), the total shunt capacitance is small and can be neglected. Therefore, a short transmission line, for the purpose of power flow and stability studies, can be represented by the simple series circuit shown in Figure 1.3.
Fig. 1.3 Equivalent circuit of a short transmission line.

Y - ∇ transformations

Fig. 1.4 Y - ∇ equivalent circuits.

The Y circuit in Figure 1.4 can be transformed into the equivalent ∇, and the node o eliminated by a Y - ∇ transformation. The impedances of the equivalent are

\[
Z_{ab} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c} = Z_a Z_b \left( \frac{1}{Z} \right) \\
Z_{bc} = Z_a Z_b + Z_b Z_c + Z_c Z_a = Z_a Z_c \left( \frac{1}{Z} \right) \\
Z_{ca} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b} = Z_c Z_a \left( \frac{1}{Z} \right)
\]

(1.15)

where

\[
\frac{1}{Z} = \frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c} = Y_a + Y_b + Y_c
\]

In terms of admittances, the above equations can be written as

\[
Y_{ab} = \frac{Y_a Y_b}{Y_a + Y_b + Y_c} \\
Y_{bc} = \frac{Y_b Y_c}{Y_a + Y_b + Y_c} \\
Y_{ca} = \frac{Y_c Y_a}{Y_a + Y_b + Y_c}
\]

(1.16)
Problem

Derive the above expressions for $Z_{ab}, Z_{bc}, Z_{ca}$

A V circuit can be transformed to an equivalent Y, where the impedances of the equivalent Y are

\[
Z_a = \frac{Z_{ab}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}
\]

\[
Z_b = \frac{Z_{ab}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}
\]

\[
Z_c = \frac{Z_{bc}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}
\]

(1.17)

Problem

Derive the above expressions for $Z_a, Z_b, Z_c$

If more than three impedances terminate on a node, the node may be eliminated by applying the general star-mesh conversion equations. However, the conversion is not reversible.

Problem

Convert the star circuit shown in Figure 1.5 into the equivalent mesh circuit and find the expressions for $Z_{ab}$, etc.

![Fig. 1.5 Star-mesh equivalent circuit.](image)

Per Unit System

In power system computations, great simplifications can be realized by employing a system in which the electrical quantities are expressed as per units of properly chosen base quantities. The per unit value of any quantity is defined as the ratio of the actual value to its base value.

Base quantities

In an electrical circuit, voltage, current, volt-ampere and impedance are so related that selection of base values of any two of them determines the base values for the remaining two. Usually, base kVA (or MVA) and base voltage in kV are the quantities selected to specify the base.

For single-phase systems (or three-phase systems where the term current refers to line current, the term voltage refers to voltage to neutral, and the term kVA refers to kVA per phase), the following formulas relate the various quantities.
Base current in amperes = \frac{\text{base kVA}}{\text{base voltage in kV}} \quad (1.18)

Base impedance in ohms = \frac{\text{base voltage in volts}}{\text{base current in amperes}}

= \frac{(\text{base voltage in kV})^2 \times 1000}{\text{base kVA}}

= \frac{(\text{base voltage in kV})^2}{\text{base MVA}} \quad (1.19)

Base power in kW = \text{base kVA}

Base power in MW = \text{base MVA}

Per unit quantity = \frac{\text{actual quantity}}{\text{base quantity}} \quad (1.20)

Occasionally a quantity may be given in percent, which is obtained by multiplying the per unit quantity by 100.

Base impedance and base current can be computed directly from three-phase values of base kV and base kVA. If we interpret base kVA and base voltage in kV to mean base kVA for the total of the three phases and base voltage from line to line, then

Base current in amperes = \frac{\text{base kVA}}{\sqrt{3} \times \text{base voltage in kV}} \quad (1.21)

Base impedance in ohms = \frac{(\text{base voltage in kV}/\sqrt{3})^2 \times 1000}{\text{base kVA}/3}

= \frac{(\text{base voltage in kV})^2 \times 1000}{\text{base kVA}}

= \frac{(\text{base voltage in kV})^2}{\text{base MVA}} \quad (1.22)

Therefore, the same equation for base impedance is valid for either single-phase or three-phase circuits. In the three-phase case line-to-line kV must be used in the equation with three-phase kVA or MVA. Line-to-neutral kV must be used with kVA or MVA per phase.

Change of bases

Per unit impedance of a circuit element = \frac{(\text{actual impedance in ohms}) \times (\text{base kVA})}{(\text{base voltage in kV})^2 \times 1000} \quad (1.23)

which shows that per unit impedance is directly proportional to the base kVA and inversely proportional to the square of the base voltage. Therefore, to change from per unit impedance on a given base to per unit impedance on a new base, the following equation applies:
Per unit \( Z_{\text{new}} = \frac{\text{per unit } Z_{\text{given}} }{ \left( \frac{\text{base kV}_{\text{given}}}{\text{base kVA}_{\text{new}}} \right)^2 \left( \frac{\text{base kVA}_{\text{new}}}{\text{base kVA}_{\text{given}}} \right) } \) (1.24)

When the resistance and reactance of a device are given by the manufacturer in percent or per unit, the base is understood to be the rated kVA and kV of the apparatus.

The ohmic values of resistance and leakage reactance of a transformer depend on whether they are measured on the high- or low-voltage side of the transformer. If they are expressed in per unit, the base kVA is understood to be the kVA rating of the transformer. The base voltage is understood to be the voltage rating of the side of the transformer where the impedance is measured.

\[
Z_{LT} = \left( \frac{kV_L}{kV_H} \right)^2 \times Z_{HT} \tag{1.25}
\]

where \( Z_{LT} \) and \( Z_{HT} \) are the impedances referred to the low-voltage and high-voltage sides of the transformer, and \( kV_L \) and \( kV_H \) are the rated low-voltage and high-voltage of the transformer, respectively.

\[
\therefore \ Z_{LT} \text{ in per unit} = \frac{(kV_L/kV_H)^2 \times Z_{HT} \times \text{kVA}}{(kV_L)^2 \times 1000} = \frac{Z_{HT} \times \text{kVA}}{(kV_H)^2 \times 1000} = Z_{HT} \text{ in per unit}
\]

A great advantage in making per unit computations is realized by the proper selection of different voltage bases for circuits connected to each other through a transformer. To achieve the advantage, the voltage bases for the circuits connected through the transformer must have the same ratio as the turns ratio of the transformer windings.

**Per unit and percent admittance**

\[
Z_{\text{ohm}} = \frac{1}{Y_{\text{mho}}}, \text{ and } Y = \frac{1}{Z}
\]

Base admittance \( Y_b = \frac{1}{Z_b} = \frac{\text{MVA}_{b}}{kV^2_b} \) (1.26)

\[
\therefore \ Y \text{pu} = \frac{Y}{Y_b} = Y \frac{kV^2_b}{\text{MVA}_{b}} = Y \frac{Z_b}{Z} = \frac{1}{Z \text{pu}} \tag{1.27}
\]

\[
Z_{\text{percent}} = Z_{\text{pu}} \times 100, \quad Y_{\text{percent}} = Y_{\text{pu}} \times 100
\]

\[
\therefore \ Y_{\text{percent}} = \frac{1}{Z_{\text{pu}}} \times 100 = \frac{10^4}{Z_{\text{percent}}} \tag{1.28}
\]

\[
P = \sqrt{3}VI \cos \phi, \quad Q = \sqrt{3}VI \sin \phi
\]
Transmission line charging is usually given in terms of total three-phase Mvar. If a transmission line has a total line charging of 50 Mvar at 485 kV, what is the per unit admittance on 100 MVA and 500 kV base?

**Problem**

Transmission line charging is usually given in terms of total three-phase Mvar. If a transmission line has a total line charging of 50 Mvar at 485 kV, what is the per unit admittance on 100 MVA and 500 kV base?

### Power Limits

Power System stability is dependent primarily upon the ability of the electrical system to interchange energy as required between the connected apparatus. It is therefore necessary to develop the fundamentals of power flow and power limit of electrical circuits.

Consider the power flow between two points along a transmission line as shown in Figure 1.6 (the shunt admittance has been neglected).

**Probem**

Derive equations (1.30) and (1.31).

From Figure 1.6

\[
P_S + jQ_S \rightarrow \frac{v_S}{v_R} \delta = \frac{v_R}{v_S} (1 + jX) \rightarrow I_S \rightarrow I_R
\]

**Fig. 1.6** Schematic of a two terminal transmission line, neglecting shunt admittance.

At the receiving end,

\[
P_R + jQ_R = \hat{V}_R \hat{I}_R^*
\]

or

\[
P_R - jQ_R = \hat{V}_R^* \hat{I}_R
\]

### Problem

Derive equations (1.30) and (1.31).

From Figure 1.6

\[
I_R = I_S = \frac{V_S \angle \delta - V_R \angle 0}{R + jX} = (V_S \angle \delta - V_R \angle 0)(G + jB)
\]

where

\[
\frac{1}{R + jX} = G + jB, \text{ the series admittance.}
\]

Substituting the above in equation (1.31),

\[
P_R - jQ_R = (V_R V_S \angle \delta - V_R^2)(G + jB) = (V_R V_S \angle \delta - V_R^2)Y \angle \theta
\]

where

---

(1) In this and subsequent chapters a “^” is used to distinguish a phasor or complex quantity from a scalar. However, this is omitted when there is no chance of confusion.
\( Y = \sqrt{G^2 + B^2} = \frac{1}{\sqrt{R^2 + X^2}} = \frac{1}{Z} \)

and
\[ \theta = \tan^{-1} \frac{B}{G} = - \tan^{-1} \frac{X}{R} = \alpha - 90^\circ \]

where
\[ \alpha = \tan^{-1} \frac{R}{X} \]

or
\[ \alpha = 90^\circ + \theta \]

For normal lines with inductive reactance \( X \), \( \alpha \) will have a small positive value.

Equating real and imaginary parts,
\[ \cos(\delta + \theta) + \frac{1}{X} \sin(\delta - \theta) = \cos \delta \sin \theta \]
\[ \sin(\delta + \theta) - \frac{1}{X} \cos(\delta - \theta) = \sin \delta \cos \theta \]

Alternatively, since \( \alpha = \theta - 90^\circ \),
\[ P_R = V_R V_S Y (\delta + \alpha) - V_R^2 Y \sin \alpha \]
and
\[ Q_R = V_R V_S Y (\delta + \alpha) - V_R^2 Y \cos \alpha \]

In terms of \( G \) and \( B \), \( P_R \) and \( Q_R \) can be written as, since \( \sin \alpha = G / Y \) and \( \cos \alpha = -B / Y \),
\[ P_R = -V_R^2 G + V_R V_S (G \cos \delta - B \sin \delta) \]
and
\[ Q_R = V_R^2 B + V_R V_S (G \sin \delta - B \cos \delta) \]

When \( R = 0 \), \( G = 0 \) and \( B = -1/X \), and equations (1.36) and (1.37) reduce to
\[ P_R = \frac{V_R V_S}{X} \sin \delta \]
and
\[ Q_R = \frac{V_R V_S}{X} \cos \delta - \frac{V_R^2}{X} \]

**Problem**

Derive expressions similar to equations (1.32) through (1.39) for \( P_S \) and \( Q_S \).

For \( R = 0 \), the maximum power that can be transferred, when \( V_R \), \( V_S \) and \( X \) are constant, is, from equation (1.38),
\[ P_{R, \text{max}} = \frac{V_R V_S}{X} \]
When $R$ is not equal to zero, the maximum power can be found by taking the partial derivative of $P_R$ with respect to $\delta$, and equating to zero. For example, from equation (1.34),

$$\frac{\partial P_R}{\partial \delta} = V_R V_S Y \cos(\delta + \alpha) = 0$$

which yields

$$\delta + \alpha = 90^\circ$$

or

$$\delta = 90^\circ - \alpha$$

and

$$P_{R\max} = V_R V_S Y - V_R^2 Y \sin \alpha = V_R V_S Y - V_R^2 G \quad (1.41)$$

**Problems**

1. Show that the maximum power at the sending end occurs at $\delta = 90^\circ + \alpha$.

2. Assuming $V_R = V_S$, find the value of the reactance ($X$) for a given value of the resistance ($R$) for maximum receiving end power. (The solution is $X = \sqrt{3} R$.)

**Circle Diagram**

Equations (1.34) and (1.35) can be rewritten as

$$P_R + V_R^2 Y \sin \alpha = V_R V_S Y \sin(\delta + \alpha) \quad (1.42)$$

$$Q_R + V_R^2 Y \cos \alpha = V_R V_S Y \cos(\delta + \alpha) \quad (1.43)$$

Squaring (1.42) and (1.43), and adding

$$\left( P_R + V_R^2 Y \sin \alpha \right)^2 + \left( Q_R + V_R^2 Y \cos \alpha \right)^2 = (V_R V_S Y)^2 \quad (1.44)$$

Equation (1.44) describes a circle with centers located at $-V_R^2 Y \sin \alpha$, $-V_R^2 Y \cos \alpha$, if $P_R$ and $Q_R$ are used as the axes of coordinates. The radius of the circle is $V_R V_S Y$. The circle diagram is shown in Figure 1.7.

**Problems**

1. Draw the sending-end circle diagram for the system shown in Figure 1.6

2. Using the circle diagram calculate $\delta$, given $P_S = 1.0$, $V_S = V_R = 1.0$, $R = 0.05$, $X = 0.3$.

It should be noted that the maximum power as given by equation (1.41), or as obtained from the circle diagram cannot be realized in practice, since this would require an uneconomically large amount of reactive power to be supplied at the receiving end in order to hold the receiving-end voltage constant. Also, above a certain level of transmitted power, the reactive support must have some sort of continuous control, such as provided by synchronous condensers or static var systems (SVS), in order to avoid voltage instability and collapse. The subject of voltage instability will be dealt with in more detail in Chapter 10.
If there are synchronous machines at both ends of a transmission line, the power limit is set by the ability of the machines to stay in synchronism with each other. This limit is called the (synchronous) stability limit. The steady-state stability limit may be defined as the maximum power that can be delivered without loss of synchronism when the load is increased gradually and the machine terminal voltages are adjusted by manual control of excitation. Similarly, the transient stability limit may be defined as the maximum power that can be delivered without loss of synchronism following a large disturbance such as a system fault and its subsequent clearing.

**Basic Power Flow Calculation**

Referring to Figure 1.6, \( P_R, Q_R, \) and the magnitude of \( V_S \) are given. It is desired to solve for \( V_R, \delta, P_S, \) and \( Q_S. \) The solution can be obtained in several ways, two of which will be illustrated here.

**Direct solution**

From (1.31)

\[
I_R = I_S = \frac{P_R - jQ_R}{V_R}
\]

\[
V_S \angle \delta = V_R + I Z = V_R + \frac{P_R - jQ_R}{V_R} (R + jX)
\]

\[
= \left( V_R + \frac{P_R R + Q_R X}{V_R} \right) + j \frac{P_R X - Q_R R}{V_R}
\]

\[
= \left( V_R + \frac{K_1}{V_R} \right) + j \frac{K_2}{V_R}
\]

(1.45)

(2) In this chapter we will follow the classical definitions of power system stability [4]. A more rigorous definition of stability is given in Chapter 2, which will be followed in subsequent chapters.
where

\[ K_1 = P_R R + Q_R X, \quad K_2 = P_R X - Q_R R \]

From (1.45)

\[ V_S^2 = \left( \frac{V_R + K_1}{V_R} \right)^2 + \left( \frac{K_2}{V_R} \right)^2 \]

which can be rearranged as

\[ \left( V_R^2 \right)^2 + \left( 2K_1 - V_S^2 \right)V_R^2 + \left( K_1^2 + K_2^2 \right) = 0 \]

From the above we obtain

\[ V_R^2 = \frac{V_S^2 - 2K_1 \pm \sqrt{\left( V_S^2 - 2K_1 \right)^2 - 4\left( K_1^2 + K_2^2 \right)}}{2} \]

\( V_R \) is obtained by taking the appropriate (positive) sign before the radical. Once \( V_R \) is known, \( \delta \) can be calculated from equation (1.45).

**Iterative procedure (Newton's method)**

Rewriting equations (1.36) and (1.37)

\[ P_R = -V_R^2 G + V_R V_S \left( G \cos \delta - B \sin \delta \right) = f_1 \left( V_R, \delta \right) \tag{1.46} \]

\[ Q_R = V_R^2 B + V_R V_S \left( G \sin \delta - B \cos \delta \right) = f_2 \left( V_R, \delta \right) \tag{1.47} \]

From (1.46) and (1.47)

\[ \Delta P_R = \frac{\partial}{\partial \delta} f_1 \left( V_R, \delta \right) \Delta \delta + \frac{\partial}{\partial V_R} f_1 \left( V_R, \delta \right) \Delta V_R \tag{1.48} \]

\[ \Delta Q_R = \frac{\partial}{\partial \delta} f_2 \left( V_R, \delta \right) \Delta \delta + \frac{\partial}{\partial V_R} f_2 \left( V_R, \delta \right) \Delta V_R \tag{1.49} \]

At the start of the iterative process

\[ \Delta P_R = P_R - f_1 \left( V_{R_0}, \delta_0 \right) \tag{1.50} \]

and

\[ \Delta Q_R = Q_R - f_2 \left( V_{R_0}, \delta_0 \right) \tag{1.51} \]

where \( V_{R_0}, \delta_0 \) are the starting values of \( V_R, \delta \).

The partial derivatives can be easily computed from (1.46) and (1.48). The changes in \( \delta \) and \( V_R \) required to improve upon the values obtained from the previous iteration are obtained by solving equations (1.48) and (1.49). For example, at the start of the iteration process \( \Delta P_R, \Delta Q_R \) and the partial derivatives at \( V_{R_0} \) and \( \delta_0 \) are computed and \( \Delta \delta_1 \) and \( \Delta V_R \) are obtained from equations (1.48) and (1.49). At the end of the first iteration

\[ \delta_1 = \delta_0 + \Delta \delta_1 \tag{1.52} \]
These updated values of $\delta$ and $V_R$ are then used in the second iteration. The iteration is continued until $\Delta P_R < \varepsilon_1$ and $\Delta Q_R < \varepsilon_2$, where $\varepsilon_1$ and $\varepsilon_2$ are the specified tolerances. The above iterative procedure can be extended to systems of any size.

**Selection of starting values for $\delta$ and $V_R$**

A good starting value for $\delta$ and $V_R$ can be obtained by using the approximations

$$G \approx 0, \quad \cos \delta \approx 1, \quad \text{and} \quad \sin \delta = \delta$$

since $G$ and $\delta$ are usually small.

To obtain the starting value of $\delta$, a further approximation, $V_R \approx V_S \approx 1.0$, can be made since the voltages are usually fairly close to unity. Using these approximations, the starting value of $\delta$ is obtained from equation (1.46) as

$$\delta_o = -P_R / B = P_R X$$  \hspace{1cm} (1.54)

In general, $V_R$ will be close to $V_S$, and therefore $V_{Ro} = V_S$ will provide a good starting value for $V_R$. If however, $Q_R$ is appreciable, a better starting value for $V_R$ can be obtained from equation (1.47) using the above approximation, as

$$V_{Ro} = V_S + Q_R / B = V_S - Q_R X$$  \hspace{1cm} (1.55)

The above method of estimating the starting values can be extended to systems of any size. This method is advisable in order to avoid divergence in the power-flow solution.

**Problem**

1. Referring to Figure 1.6, the following quantities are given:

   $P_R = 0.85$, $Q_R = 0.35$, $V_S = 1.15$

   Calculate $\delta$, $V_R$, $P_S$ and $Q_S$.

2. Referring to Figure 1.6, the following quantities are given:

   $P_R = 0.8$, $Q_R = 0.4$, $V_R = 1.0$

   Calculate $\delta$, $V_S$, $P_S$ and $Q_S$.

**Steady State Stability**

Consider a cylindrical rotor synchronous generator connected to a large power system (infinite bus) operating at constant speed with constant field current and no saturation, as shown in Figure 1.8.

Fig. 1.8 A cylindrical rotor synchronous generator connected to infinite bus and phasor diagram
$r_a$ and $x_d$ are the armature resistance and synchronous reactance, respectively, and $E$ is the internal machine voltage or the voltage corresponding to field excitation.

Since $E$ is produced by the field flux, it is associated with the rotor position. The power output of the machine is given by

$$P = \frac{E^2 \sin \alpha}{\sqrt{r_a^2 + x_d^2}} + \frac{EV}{\sqrt{r_a^2 + x_d^2}} \sin(\delta - \alpha) \quad (1.56)$$

where

$$\alpha = \tan^{-1} \left( \frac{r_a}{x_d} \right)$$

Since $r_a$ is usually small, it can be neglected. Then

$$P = \frac{EV}{x_d} \sin \delta \quad (1.57)$$

The maximum power output for fixed $E$ and $V$ is

$$P_{\text{max}} = \frac{EV}{x_d} \quad (1.58)$$

This is termed the pull-out power of a synchronous generator against an infinite bus at constant field excitation, since any attempt to increase the power output further will result in a loss of synchronism.

Equation (1.56) represents the power angle characteristic of a cylindrical rotor synchronous machine connected directly to an infinite bus.

**Problem**

Calculate the power angle characteristic of a cylindrical rotor generator connected through an external reactance to an infinite bus of voltage 1.0 pu, as shown in Figure 1.9. $E$ is such that 1.0 pu voltage is obtained at the terminal at $P = 1.0$ pu.

Fig. 1.9  A cylindrical rotor synchronous generator connected to an infinite bus through an external reactance.

Steady state stability is the stability of the system under conditions of gradual or relatively slow changes in load. The load is assumed to be applied at a rate which is slow when compared either with the natural frequency of oscillation of the major parts of the system or with the rate of change of field flux in the machine in response to the change in loading.

We have already discussed the maximum power at constant field current against an infinite bus.

**Steady state stability limit**

The maximum power which a generator can deliver depends upon the field excitation of the generator. Within limit, the excitation depends upon the system operating condition. Generators
FUNDAMENTALS OF POWER FLOW AND POWER LIMITS

will usually have their excitations adjusted to hold their terminal voltages at some predetermined values. The terminal voltage is held either manually or with an automatic voltage regulator.

If the load changes occur slowly or gradually so that steady state conditions may be assumed to exist, the field excitation will change slowly in order to adjust the terminal or system voltage to the prescribed value. The steady state stability limit of a generator or system can be defined as the maximum power that can be transmitted for a slow change in load, the load change occurring slowly enough to allow for a similar change in excitation to bring the terminal voltage back to normal after each successive load change. It is important to note that it has been assumed that the control of excitation is such as to correct the voltage change after each small load change has occurred. Therefore, this is a stability limit for an infinitesimal change in load with constant field current. If the change in excitation is assumed to be initiated immediately following the change in load (as happens with automatic voltage regulator action), the stability limit under such conditions may be termed dynamic stability limit. The dynamic limit will, in general, be higher than the steady state stability limit as defined above. However, the dynamic stability limit is dependent on automatic voltage regulator operation.

Example

Consider the system and its phasor diagram shown in Figure 1.10. Operating conditions are such that 1.0 pu voltage is maintained at both the infinite bus and the generator terminals. Saturation is neglected. At zero initial load, the generator excitation is also unity. Therefore

\[ P = \frac{E V}{x_d + x_e} \sin \delta = \frac{1.0 \times 1.0}{2.0} \sin \delta = 0.5 \sin \delta \]

As the initial power increases, the initial excitation also increases. For example, at initial power of 0.5, the excitation voltage \( E \) can be calculated, that satisfies the given condition of 1.0 pu voltage at the generator terminal and the infinite bus, from the phasor diagram. \( E \) is found to be, \( E = 1.24 \).

**Problem:** Verify the above value of \( E \) for \( P = 0.5 \).

Therefore, the power angle relation is

\[ P = \frac{1.0 \times 1.24}{2.0} \sin \delta = 0.62 \sin \delta \]

**Problem**

Calculate the power angle relations for initial power levels of 0.70, 0.75, 0.80, 0.85 and 0.90.

It will be found that at \( P = 0.85 \), the initial power corresponds to an initial angular displacement of 90°, as shown in Figure 1.11.
The steady state stability limit corresponds to the point at which, if the receiving system requires a small additional load $\Delta P$, the generator becomes just unable to deliver it without a change in the field excitation. Therefore, at the limit $dP/d\delta = 0$. This is the condition at $P = 0.85$ and $\delta = 90^\circ$, where $dP/d\delta$ is measured on curve c of Figure 1.11. If a higher load is taken, the initial angle is greater than 90°, and $dP/d\delta$ becomes negative, when measured along the corresponding power angle curve for constant field excitation.

**Steady state stability criterion**

For the simple case of a single generator connected to an infinite bus, the condition for steady state stability is simply that $dP/d\delta$ be positive at the given operating point as shown above. The steady state power limit corresponds to the maximum power that can be delivered up to the critical point where $dP/d\delta$ changes from a positive to a negative value. $dP/d\delta$ is called the steady state synchronizing power coefficient since it indicates the rate at which the steady or sustained power changes with changes in electrical angular displacement.

It should be noted that the synchronizing power coefficient can increase considerably by the action of automatic voltage regulators. In a simplified analysis, the effect of automatic voltage regulators can be approximately accounted for by suitably adjusting the equivalent machine reactance. However, as will be seen later, the usual form of instability in the steady state, under automatic voltage control, is due to lack of damping rather than synchronizing power. Nevertheless, the concept of steady state stability limit as discussed above is useful in developing a proper understanding of the subject of power system stability.

**Steady state stability of a two-machine system**

Many steady state stability problems resolve themselves into a two-machine problem. For example, a large remotely located generating plant delivering power to the main system can often be treated as a two-machine problem with the system represented as an equivalent machine -- in effect as a large equivalent synchronous motor. A value of reactance corresponding to the total three-phase short circuit current contributed by the receiving system may be used as the
equivalent reactance of the receiving-end system. A graphical method may be utilized to determine the stability limit.

For the simplified two-machine case shown in Figure 1.12, the steady-state stability limit occurs when maximum power is obtained at the receiving end, i.e., when the total angle between the equivalent synchronous machine groups corresponds to the total impedance angle.

\[
\delta_{12} = \tan^{-1}\left(\frac{x_{12}}{r}\right), \quad \text{where} \quad x_{12} = x_g + x + x_m
\]

(Note that \(dP/d\delta\) becomes negative first at the receiving end, which determines the steady state stability limit.)

A further condition is imposed by the operating conditions of the system, i.e., the magnitudes of the terminal voltages of the equivalent synchronous generator and motor at the sending and receiving ends, respectively. Using the graphical method, the stability limit can be determined directly without resorting to a trial and error procedure. The method is as follows:

Select a reference direction for current \(I\). Draw phasors \(jI x_g\), \(I(r + jx)\), and \(jI x_m\) to any convenient scale as shown in Figure 1.13. In this figure, 1 and 2 refer to the points in the generator and motor at which voltages are maintained when the final increment of load is added; a and b are the terminals of the generator and the motor, respectively.

To locate the origin of the phasor diagram: Connect points 1 and 2 by a straight line and draw a perpendicular at the mid-point M of this line. Extend the line 2b until it intersects this perpendicular at point P. With P as center draw a circle passing through points 1 and 2 (i.e., with
a radius P2). Any point O on the larger of the two arcs (as shown on Fig.1.13) satisfies the condition that \( \delta_{12} = \theta = \tan^{-1}(x_{12}/r) \) (verify this from the geometry of Fig. 1.13).

In order to satisfy the voltage condition, the point O must be so located on the arc that the distances Oa and Ob are in the ratio of the terminal voltages \( V_a \) and \( V_b \). This may be done by determining a series of points with a and b as centers and radii in the ratio \( V_a/V_b \). The intersection of the locus of such points with the arc will completely define the location of the origin O of the phasor diagram. The scale of the diagram will be determined by the magnitudes of the specified voltages \( V_a \) and \( V_b \) at the generator and motor terminals.

If a \( \pi \) circuit is used to represent the transmission line as shown in Figure 1.14, the system can be reduced to the form of Fig. 1.12 by using the Thevenin equivalent shown in Figure 1.15.

![Fig. 1.14 Equivalent two-machine system including the effect of shunt capacitance.](image)

![Fig. 1.15 Thevenin equivalent of the system of Fig. 1.14.](image)

The same procedure as that used for the system of Figure 1.12 can therefore be used.

**Problem**

Verify the values of \( E'_1, E'_2, x'_g \), and \( x'_m \).

If the line resistance is neglected and equal sending- and receiving-end machine terminal voltages are assumed, Figure 1.13 simplifies to Figure 1.16. At the steady state stability limit \( \delta_{12} \) will be equal to 90°.

![Fig. 1.16 Simplification of Fig. 1.13 neglecting \( r \).](image)
Problem
Show that the maximum power for the case shown in Figure 1.16 is given by
\[
P_{\text{max}} = \frac{\sqrt{(x_g + x/2)(x_m + x/2)}}{(x/2)^2 + (x_g + x/2)(x_m + x/2)}
\]  
(1.59)

(Hint: Use the relations \(OM^2 = 1M \times 2M\), and \(P_{\text{max}} = E_mI\))

General Network Equations
An electrical network can be analyzed by expressing the electrical characteristics of the individual network elements and their interconnection relationship in terms of mathematical equations. The network performance equations can be written in nodal, loop or branch form. In large power system analyses it is both convenient and customary to use the nodal analysis approach. In the nodal approach, the bus admittance matrix formulation is generally computationally more expedient.

Nodal formulation in terms of bus admittance matrix
The general expression for the current toward node \(k\) of a network having \(n\) independent nodes (i.e., \(n\) buses other than the neutral which is taken as a reference) is
\[
I_k = \sum_{i=1}^{n} Y_{ki} V_i
\]  
(1.60)

For example, for a four bus system the node equations are
\[
\begin{align*}
I_1 &= Y_{11} V_1 + Y_{12} V_2 + Y_{13} V_3 + Y_{14} V_4 \\
I_2 &= Y_{21} V_1 + Y_{22} V_2 + Y_{23} V_3 + Y_{24} V_4 \\
I_3 &= Y_{31} V_1 + Y_{32} V_2 + Y_{33} V_3 + Y_{34} V_4 \\
I_4 &= Y_{41} V_1 + Y_{42} V_2 + Y_{43} V_3 + Y_{44} V_4
\end{align*}
\]  
(1.61)

which can be written in matrix form as
\[
I = Y V
\]  
(1.62)

where \(Y\) is the network bus admittance matrix, and \(I\) and \(V\) are the current and voltage vectors, respectively.

The admittances \(Y_{11}, Y_{22}\), etc. are called the self or driving-point admittances at the nodes, and each equals the sum of all the admittances terminating on the node identified by the repeated subscripts. The other admittances are the mutual or transfer admittances, and each equals the negative of the admittance connected between the nodes identified by the double subscripts. For example, if bus 1 is connected to the other buses in the network as shown in Figure 1.17, then
\[
Y_{11} = y'_{10} + y'_{12} + y'_{14} = G_{11} + jB_{11}
\]

where
\[
y'_{10} = \frac{1}{r_{10} + jx_{10}}, \quad y'_{12} = \frac{1}{r_{12} + jx_{12}}, \quad \text{etc.}
\]
Problem

Derive equation (1.60)

General equations for real and reactive power

Real and reactive power at node $i$ are given by

$$P_i - jQ_i = V_i^* I_i = V_i^* \sum_{k=1}^{n} Y_{ik} V_k$$

which, in expanded form, can be written as

$$P_i - jQ_i = V_i^* \left[ Y_{ii} V_i + Y_{i2} V_2 + \cdots + Y_{in} V_n \right]$$

$$= V_i V_i Y_{ii} e^{-j(\delta_i - \theta_i)} + V_i V_2 Y_{i2} e^{-j(\delta_{i2} - \theta_{i2})} + \cdots$$

$$+ V_i^2 Y_{ii} e^{j\theta_i} + \cdots + V_i V_n Y_{in} e^{-j(\delta_{in} - \theta_{in})}$$

(1.64)

where the $\delta$’s and $\theta$’s are the voltage and admittance angles, respectively.

Equating real and imaginary parts, noting that $e^{-j(\delta_{in} - \theta_{in})} = \cos(\delta_{in} - \theta_{in}) - j \sin(\delta_{in} - \theta_{in})$,

$$P_i = V_i V_i Y_{ii} \cos(\delta_{i1} - \theta_{i1}) + V_i V_2 Y_{i2} \cos(\delta_{i2} - \theta_{i2}) + \cdots$$

$$+ V_i^2 Y_{ii} \cos \theta_{i1} + \cdots + V_i V_n Y_{in} \cos(\delta_{in} - \theta_{in})$$

which can be written as

$$P_i = \sum_{k=1}^{n} V_i V_k Y_{ik} \cos(\delta_{ik} - \theta_{ik})$$

(1.65)

and

$$Q_i = V_i V_i Y_{ii} \sin(\delta_{i1} - \theta_{i1}) + V_i V_2 Y_{i2} \sin(\delta_{i2} - \theta_{i2}) + \cdots$$

$$- V_i^2 Y_{ii} \sin \theta_{i1} + \cdots + V_i V_n Y_{in} \sin(\delta_{in} - \theta_{in})$$

which can be written as
\[ Q_i = \sum_{k=1}^{n} V_i V_k y_{ik} \sin (\delta_{ik} - \theta_{ik}) \tag{1.66} \]

Alternatively, equation (1.64) can be written as

\[
P_i - jQ_i = (G_{i1} + jB_{i1}) V_i V_1 e^{-j\delta_{i1}} + (G_{i2} + jB_{i2}) V_i V_2 e^{-j\delta_{i2}} + \ldots + (G_{in} + jB_{in}) V_i V_n e^{-j\delta_{in}} \tag{1.67} \]

Equating real and imaginary parts, noting that \( e^{-j\delta_n} = \cos \delta_{in} - j \sin \delta_{in}, \)

\[
P_i = V_i V_1 G_{i1} \cos \delta_{i1} + V_i V_2 G_{i2} \cos \delta_{i2} + \ldots + V_i^2 G_{i1} + \ldots + V_i V_n G_{in} \cos \delta_{in} + V_i V_1 B_{i1} \sin \delta_{i1} + V_i V_2 B_{i2} \sin \delta_{i2} + \ldots + V_i V_n B_{in} \sin \delta_{in} \]

which can be written as

\[
P_i = \sum_{k=1}^{n} V_i V_k (G_{ik} \cos \delta_{ik} + B_{ik} \sin \delta_{ik}) \tag{1.68} \]

and

\[
Q_i = V_i V_1 G_{i1} \sin \delta_{i1} + V_i V_2 G_{i2} \sin \delta_{i2} + \ldots + V_i^2 G_{i1} + \ldots + V_i V_n G_{in} \sin \delta_{in} - V_i V_1 B_{i1} \cos \delta_{i1} - V_i V_2 B_{i2} \cos \delta_{i2} - \ldots - V_i^2 B_{i1} - \ldots - V_i V_n B_{in} \cos \delta_{in} \]

which can be written as

\[
Q_i = \sum_{k=1}^{n} V_i V_k (G_{ik} \sin \delta_{ik} - B_{ik} \cos \delta_{ik}) \tag{1.69} \]

**Elimination of network nodes by matrix method**

Consider the two-machine three-bus system shown in Figure 1.18. Nodes 3 and 5 correspond to the terminal buses of machines 1 and 2, nodes 1 and 2 correspond to the machine internal buses. Note that, including the machine internal buses, the total number of nodes in the system is five. We wish to eliminate nodes 3, 4 and 5.

![Schematic of a three-bus two-machine system](image)
The current and voltage relationship can be written in matrix form as

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{15} \\
Y_{21} & Y_{22} & \cdots & & \\
Y_{31} & \cdots & Y_{33} & \cdots & \\
\vdots & \ddots & \cdots & \ddots & \\
Y_{51} & \cdots & \cdots & Y_{55}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5
\end{bmatrix}
\]

(1.70)

After partitioning as shown, (1.70) can be written as

\[
\begin{bmatrix}
I \\
I^T
\end{bmatrix} =
\begin{bmatrix}
Y_1 & Y_2 \\
Y_3 & Y_4
\end{bmatrix}
\begin{bmatrix}
V \\
V^T
\end{bmatrix}
\]

(1.71)

Since buses 3, 4 and 5 are passive nodes, the elements of the vector \( I^T \) are zero. Therefore, eliminating \( V^T \) from (1.71) (see Appendix A),

\[
I = \begin{bmatrix} Y_1 - Y_2 Y_4^{-1} Y_3 \end{bmatrix} V = YV
\]

(1.72)

The reduced admittance matrix \( Y \) is given by

\[
Y = Y_1 - Y_2 Y_4^{-1} Y_3
\]

(1.73)

**Application of the Steady-State Stability Criterion to an n Machine System**

For a single generator connected to an infinite bus a steady-state stability criterion has been derived earlier. According to this criterion, for steady-state stability, the synchronizing power coefficient \( dP/d\delta \) is required to be positive at the given operating point. In applying this criterion to a multi-machine system, the distribution of load among generators in the system due to a small incremental load change has to be assumed. A conservative steady state limit is obtained if it is assumed that an increase in shaft load is taken by only one machine, with the added generation being taken by one other machine, and with all the remaining machines operating at constant power. If this is done in turn with all the machines in the system, and in all cases the synchronizing power coefficient turns out to be positive, the system is stable at the assumed operating condition. Usually the two machines having the largest angular displacement are taken as the loaded and balancing machines, since these two are the least stable. All the other machines are treated as constant power machines. Numbering the loaded and balancing machines by 1 and \( n \), \( dP/d\delta_{1n} \) and \( dP/d\delta_{n1} \) should both be positive for stability. These can be calculated as follows:

For an assumed load, under given operating conditions, the internal voltages of all the machines are calculated. The network is then reduced, retaining only the machine internal buses. The power output of machine \( i \) is given by equation (1.68), repeated here for convenience.

\[
P_i = \sum_{k=1}^{n} V_k V_{ik} \left( G_{ik} \cos \delta_{ik} + B_{ik} \sin \delta_{ik} \right) \quad i = 1, 2, \cdots n
\]

(1.68)

where \( G_{ik}, \) etc. correspond to the reduced admittance matrix.
With \( dP_2 = dP_3 = \cdots = dP_{n-1} = 0 \),

\[
\begin{align*}
\frac{dP_1}{d\delta_{1n}} &= \frac{\partial P_1}{\partial \delta_{1n}} + \frac{\partial P_1}{\partial \delta_{2n}} \frac{d\delta_{2n}}{d\delta_{1n}} + \cdots + \frac{\partial P_1}{\partial \delta_{n-1,n}} \frac{d\delta_{n-1,n}}{d\delta_{1n}} \\
0 &= \frac{\partial P_2}{\partial \delta_{1n}} + \frac{\partial P_2}{\partial \delta_{2n}} \frac{d\delta_{2n}}{d\delta_{1n}} + \cdots + \frac{\partial P_2}{\partial \delta_{n-1,n}} \frac{d\delta_{n-1,n}}{d\delta_{1n}} \\
&\quad \cdots \quad \cdots \quad \cdots \\
\frac{dP_n}{d\delta_{1n}} &= \frac{\partial P_n}{\partial \delta_{1n}} + \frac{\partial P_n}{\partial \delta_{2n}} \frac{d\delta_{2n}}{d\delta_{1n}} + \cdots + \frac{\partial P_n}{\partial \delta_{n-1,n}} \frac{d\delta_{n-1,n}}{d\delta_{1n}}
\end{align*}
\]

which can be written in matrix form

\[
\begin{bmatrix}
1 & -\frac{\partial P_1}{\partial \delta_{2n}} & -\frac{\partial P_1}{\partial \delta_{3n}} & \cdots & -\frac{\partial P_1}{\partial \delta_{n-1,n}} \\
0 & -\frac{\partial P_2}{\partial \delta_{2n}} & -\frac{\partial P_2}{\partial \delta_{3n}} & \cdots & -\frac{\partial P_2}{\partial \delta_{n-1,n}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -\frac{\partial P_n}{\partial \delta_{2n}} & -\frac{\partial P_n}{\partial \delta_{3n}} & \cdots & -\frac{\partial P_n}{\partial \delta_{n-1,n}}
\end{bmatrix}
\begin{bmatrix}
\frac{dP_1}{d\delta_{1n}} \\
\frac{dP_2}{d\delta_{1n}} \\
\vdots \\
\frac{dP_n}{d\delta_{1n}}
\end{bmatrix} =
\begin{bmatrix}
\partial P_1 \\
\partial P_2 \\
\vdots \\
\partial P_n
\end{bmatrix}
\]

From equation (1.74) \( dP_1/d\delta_{1n} \) and \( dP_n/d\delta_{1n} \) can be computed, since all the elements in the matrix and the right-hand column vector are known.

\( \frac{\partial P_i}{\partial \delta_{1n}} \), \( \frac{\partial P_i}{\partial \delta_{2n}} \), etc. are obtained from equation (1.68), noting that \( \delta_{12} = \delta_{1n} - \delta_{2n} \), etc. These are given by

\[
\frac{\partial P_i}{\partial \delta_{1n}} = \sum_{j=1 \atop j \neq i}^n V_i V_j \left( -G_{ij} \sin \delta_{ij} + B_{ij} \cos \delta_{ij} \right) \quad i = 1, 2, \cdots n-1
\]

\[
\frac{\partial P_i}{\partial \delta_{jn}} = V_i V_j \left( G_{ij} \sin \delta_{ij} - B_{ij} \cos \delta_{ij} \right) \quad i = 1, 2, \cdots n \\
\quad \quad \quad j = 1, 2, \cdots n-1, \quad i \neq j
\]

**Problems**

1. Verify equations (1.74), (1.75) and (1.76).

2. Find the expression for \( dP_1/d\delta_{13} \) for a three-machine system. Neglect all conductances.

**Network Calculation Using Bus Impedance Matrix**

Although the use of the bus admittance matrix is generally desirable in the solution of power system problems, there are situations when the use of the bus impedance matrix can be attractive. For example, the use of the bus impedance matrix greatly simplifies short circuit computations. Also, the bus impedance matrix is ideally suited for rapid, approximate analysis of certain system
contingencies, e.g., loss of a transmission line, loss of a generator, etc. In the past, the bus impedance matrix also found extensive use in power flow and stability calculations.

The bus impedance matrix can be obtained by a direct inversion of the bus admittance matrix. For example, from equation (1.62), one can obtain

$$V = Y^{-1}I = ZI$$

(1.77)

where $Z$ is the bus impedance matrix.

The direct inversion is not, however, practical when dealing with large systems typically containing several hundred and frequently well over a thousand buses. A direct inversion can be avoided by expressing the admittance matrix as a product of two triangular (lower and upper) matrices and utilizing a process called backward and forward substitution. The bus admittance matrix is extremely sparse. In a typical power system a particular bus is connected, on an average, to three other buses and rarely to more than four, so that a typical row or column of the bus admittance matrix has four non-zero elements. By optimally re-ordering the rows and columns before factoring, sparsity can be maintained in the resulting triangular matrices. Since only the non-zero elements need be stored and used in the computation, great savings in computer time and storage can, therefore, be realized. Various schemes for optimally ordered triangular factorization are available.

In contrast, the bus impedance matrix is generally full, i.e., there are very few zero elements in the matrix. If, however, only a small number of elements are required in a particular study, these can be selectively calculated from the bus admittance matrix using the above technique, thereby avoiding the necessity to handle the complete matrix.

For small and medium sized systems it can be more convenient to build the bus impedance matrix directly from network and system parameters. Starting with one bus selected arbitrarily, the building of the bus impedance matrix proceeds by adding the remaining buses and lines, one at a time, and applying one of the three basic procedures as new buses and lines are added. Also, once the bus impedance matrix of the original network has been formed, any modification to it following a change in the network (line tripping, line closing, etc.) can be easily and efficiently performed by applying these procedures. These procedures will now be described.

(i) **Adding a new bus to the network**

Consider a new bus, numbered $n+1$, being added to bus $i$ of the network of $n$ buses as shown in Figure 1.19.

![Fig. 1.19 Illustration of the addition of a new bus to an existing network.](image)
Injecting a unit current at the new bus \( n+1 \) is equivalent to injecting a unit current at bus \( i \) as far as the existing network is concerned. Therefore, in accordance with equation (1.78), the changes in voltages at buses \( 1, 2, ..., n \) are \( Z_{ii}, Z_{i2}, ..., Z_{in} \), respectively.

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n \\
V_{n+1}
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1n} \\
Z_{21} & \cdots & \cdots & Z_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{n1} & \cdots & \cdots & Z_{nn}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n
\end{bmatrix}
\]  
(1.78)

The change in voltage at bus \( n+1 \) is, from Figure 1.19, \( Z_{ii} + z \), where \( z \) is the impedance of the line between buses \( i \) and \( n+1 \).

The change in voltage at the new bus \( n+1 \) due to a unit injected current at any of the buses \( 1 \) to \( n \) will be the same as the change in voltage at bus \( i \) due to the same injected current. Therefore, from the principle of superposition, the modified impedance matrix following the addition of the new bus will be as shown in equation (1.79).

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n \\
V_{n+1}
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1n} & Z_{1i} \\
Z_{21} & \cdots & \cdots & Z_{2n} & Z_{2i} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Z_{n1} & \cdots & \cdots & Z_{nn} & Z_{ni}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n \\
I_{n+1}
\end{bmatrix}
\]  
(1.79)

(ii) Adding a new line between two existing buses

Consider a new line being added between buses \( i \) and \( j \) as shown in Figure 1.20. When a new line of impedance \( z \) is added between buses \( i \) and \( j \), a current \( I \) will flow on the line as shown in the figure. This current can be obtained from a knowledge of the Thevenin impedance \( Z_{Th} \). \( Z_{Th} \) can be obtained by injecting a unit positive current at \( i \) and a unit negative current at \( j \) and noting the change in voltage between buses \( i \) and \( j \). Following this procedure, \( Z_{Th} \) is obtained from equation (1.78) as

\[
Z_{Th} = (Z_{ii} - Z_{ij}) - (-Z_{ji} + Z_{jj}) = Z_{ii} + Z_{jj} - 2Z_{ij}
\]  
(1.80)

for \( Z_{ij} = Z_{ji} \)

Therefore, the current \( I \) flowing in the new line between buses \( i \) and \( j \) is given by
\[ I = \frac{V_i - V_j}{Z_{th} + z} \]

which yields, using (1.78)

\[ I = \frac{\sum_{k=1}^{n} (Z_{ik} - Z_{jk})I_k}{Z_{th} + z} \]  \hspace{1cm} (1.81)

Equation (1.81) can be written as

\[ \sum_{k=1}^{n} (Z_{ik} - Z_{jk})I_k + (Z_{th} + z)(-I) = 0 \]  \hspace{1cm} (1.82)

From the viewpoint of the existing network, adding a new line between buses \( i \) and \( j \) is equivalent to injecting a current \(-I\) at \( i \) and \(+I\) at \( j \). The change in voltage at bus \( k \), due to the new line, is then

\[ \Delta V_k = (-Z_{ki} + Z_{kj})I \hspace{1cm} k = 1, 2, \cdots n \]  \hspace{1cm} (1.83)

Therefore, the new voltage at bus \( k \) is given by

\[ V_k = V_{ko} + \Delta V_k \]

which yields, using (1.78)

\[ V_k = \sum_{l=1}^{n} Z_{kl}I_l + (Z_{ki} - Z_{kj})(-1) \hspace{1cm} k = 1, 2, \cdots n \]  \hspace{1cm} (1.84)

Equations (1.82) and (1.84) can be combined and written in matrix form as

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n \\
V_{ko}
\end{bmatrix}
= 
\begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1n} & Z_{1i} - Z_{1j} \\
Z_{21} & \ddots & \ddots & \vdots & Z_{2i} - Z_{2j} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
Z_{ni} & \cdots & \cdots & Z_{nn} & Z_{ni} - Z_{nj} \\
Z_{i1} - Z_{j1} & Z_{i2} - Z_{j2} & \cdots & Z_{in} - Z_{jn} & Z_{th} + z \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n \\
-I
\end{bmatrix}
\]  \hspace{1cm} (1.85)

The modified \( Z \) matrix is therefore obtained by eliminating the last row and column of the matrix of equation (1.85). For example, the \( lk \)th element of the modified matrix will be given by

\[ Z'_{ik} = Z_{ik} - \frac{(Z_{li} - Z_{lj})(Z_{ik} - Z_{jk})}{Z_{th} + z} \]  \hspace{1cm} (1.86)

Note: Equations (1.81) and (1.83) can be used to calculate the changes in bus voltages following a line trip. A line trip is simulated by adding a line of equal impedance and of opposite sign in parallel with the existing line. Therefore, in order to simulate a line trip, simply put

\[ z = - \text{(impedance of the line to be tripped)} \]

(iii) **Adding a new line between an existing bus and neutral (reference bus).**

Following the same argument as presented in (ii), we arrive at an equation similar to equation (1.85), as shown below:
\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n \\
0
\end{bmatrix} =
\begin{bmatrix}
Z_{i1} & Z_{i2} & \cdots & Z_{in} & Z_{ii} \\
Z_{21} & \ddots & \cdots & \vdots & Z_{2i} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
Z_{n1} & \cdots & Z_{nn} & Z_{ni} & I_n \\
Z_{i1} & Z_{i2} & \cdots & Z_{in} & Z_{Th} + z
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n \\
-I_j
\end{bmatrix}
\]

(1.87)

where \( Z_{Th} = Z_{ii} \), \( i \) is the bus where the new line is connected.

The modified bus impedance matrix is obtained after eliminating the last row and column from equation (1.87). For example, the \( l^\text{th} \) element of the modified matrix will be given by

\[
Z'_{ik} = Z_{ik} - \frac{Z_{il}Z_{ik}}{Z_{Th} + z}
\]

(1.88)

The building of the network bus impedance matrix proceeds as follows:

First select a reference bus, usually the neutral (ground). Pick any bus connected to the reference bus. This bus is numbered 1. The impedance matrix now has one element whose value is the impedance between the bus numbered 1 and the reference (numbered zero). As new buses and lines are added, the matrix is modified in accordance with the procedures described in steps (i), (ii), or (iii). The process is continued until all the buses and lines have been included in the matrix.

**Calculation of Short-Circuit Current**

Consider a three-phase fault applied at bus \( i \) through an impedance \( z \) as shown in Figure 1.21.

![Fig. 1.21 Illustration of the calculation of three-phase short circuit current.](image)

Between bus \( i \) and ground, the Thevenin impedance is \( Z_{ii} \). Therefore, the short circuit current is given by

\[
I_{sh} = \frac{V_i}{Z_{ii} + z}
\]

(1.89)

The change in voltage at bus \( k \) due to this current is given by

\[
\Delta V_k = \frac{-Z_{ik}V_i}{Z_{ii} + z} \quad k = 1, 2, \cdots n
\]

(1.90)

The change in the current flow on a line connecting buses \( k \) and \( l \) is given by

\[
\Delta I_{kl} = \frac{\Delta V_k - \Delta V_l}{z_{kl}}
\]

(1.91)

Note: Equation (1.89) can also be derived as follows:
When bus $i$ is shorted through an impedance $z$, the voltage of bus $i$ becomes $Iz$, where $I$ is the short-circuit current flowing from the bus to ground. Therefore, the change in voltage at bus $i$ is

$$
\Delta V_i = Iz - V_i
$$

(1.92)

The change in voltage at bus $i$ is also obtained from equation (1.78) due to an injected current $-I$ at $i$, given by

$$
\Delta V_i = -Z_{ii}I
$$

(1.93)

Equation (1.89) follows from (1.92) and (1.93).

In many short circuit studies of very large systems, the short circuit information pertaining to a part of the system is only needed. In such cases it is unnecessary to build and work with the impedance matrix of the entire system. During the matrix building process, in the addition of a new bus to an existing one, the only elements that take an active part in the matrix modifications are the elements of the row and column corresponding to the existing bus. When a new line is added, the only rows and columns that take an active part in the matrix modification are the rows and columns corresponding to the buses being connected by the new line. Therefore, when all lines have been connected to a bus, the elements of the matrix in the row and column corresponding to that bus will never be required as active participants in the matrix modification. If the bus is not required in the short circuit analysis, the row and column corresponding to the bus may be deleted. When all lines have been processed and the rows and columns corresponding to the buses not required for the short circuit analysis have been deleted following the above procedure, the remaining matrix will be the impedance matrix corresponding to the study area, with the effect of the rest of the system properly taken into account.

**Representation of Tap-changing Transformers and Phase-shifting Transformers (Power Angle Regulators) in Power Flow and Stability Studies**

Consider a transformer with off-nominal tap ratio $a$ (per unit) between buses $p$ and $q$ as shown in Figure 1.22. The impedance $Z$ represents the sum of the transformer leakage impedance and any other series impedance between buses $p$ and $q$. $a$ can be real or complex.

![Fig. 1.22 Schematic of tap-changing transformer with an off-nominal tap ratio $a$.](image)

From Figure 1.22

$$
V_q = aV_r
$$

or

$$
V_r = V_q / a
$$

(1.94)

Equating complex powers on each side of the ideal transformer

$$
V_rI_p^* = aV_rI_p^* \quad \text{or} \quad I = I_p / a^*
$$

which yields

$$
I_q = -I_p / a^*
$$

(1.95)
We have
\[
\frac{V_p - V_r}{Z} = I_p
\] (1.96)

From (1.94) and (1.96), we have
\[
I_p = \frac{V_p}{Z} - \frac{V_q}{aZ}
\] (1.97)

From (1.95) and (1.97), we have
\[
I_q = -\frac{V_p}{a^2Z} + \frac{V_q}{|a|^2Z}
\] (1.98)

Equations (1.97) and (1.98) can be written in matrix form as
\[
\begin{bmatrix}
I_p \\
I_q
\end{bmatrix} = \begin{bmatrix}
\frac{1}{Z} & -1 \\
\frac{1}{aZ} & 1
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q
\end{bmatrix} = \begin{bmatrix}
Y & -\frac{Y}{a} \\
-\frac{Y}{a^2} & \frac{Y}{|a|^2}
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q
\end{bmatrix}
\] (1.99)

where
\[
Y = 1/Z
\]

If \( a \) is real, as in the case of in-phase tap-changing transformer, (1.99) reduces to
\[
\begin{bmatrix}
I_p \\
I_q
\end{bmatrix} = \begin{bmatrix}
\frac{1}{Z} & -1 \\
\frac{1}{aZ} & 1
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q
\end{bmatrix} = \begin{bmatrix}
Y & -Y \\
-\frac{Y}{a} & \frac{Y}{a^2}
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q
\end{bmatrix}
\] (1.100)

From (1.100), the equivalent circuits of Figure 1.23 follow.

![Impedance and Admittance Diagrams](image)

**Fig. 1.23** Equivalent circuits of in-phase tap-changing transformer.

**Effect of phase-shift on power flow**

Consider the example shown in Figure 1.24

![Phase shift effect](image)

**Fig. 1.24** Illustration of the effect of phase shift on power flow.
We have

\[ P_p - jQ_p = V_p^* I_p \]  

(1.101)

Substituting \( I_p \) from (1.99) into (1.101), we have

\[ P_p - jQ_p = V_p^* \left( Y V_p - \frac{Y}{a} V_q \right) \]  

(1.102)

Also

\[ a = e^{j\phi}, \quad V_p = |V_p| e^{j\theta} \]

Assuming that the impedance is fully reactive,

\[ Y = \frac{1}{jX} = \frac{1}{X} e^{-jn/2} \]

Substituting the expressions for \( a, V_p, \) and \( Y \) in (1.102), we have

\[ P_p - jQ_p = \frac{V_p^2}{X} e^{-j\pi/2} - \frac{V_p V_q}{X} e^{-j(\pi/2 + \theta + \phi)} \]  

(1.103)

Equating the real and imaginary parts, we have

\[ P_p = \frac{V_p V_q}{X} \sin (\theta + \phi) \]  

(1.104)

\[ Q_p = \frac{V_p^2}{X} - \frac{V_p V_q}{X} \cos (\theta + \phi) \]  

(1.105)

**Example 1**

A phase shifting transformer with a phase shift \( \phi \) is connected between buses p and q as shown in Figure 1.25a. The phasor relationship of the voltages is shown in Figure 1.25b.

Fig. 1.25 Phase shifting transformer with phase shift \( \phi \) connected between buses p and q, and phasor diagram.

**Example 2**

Consider the arrangement shown in Figure 1.26a, where an ideal phase shifter with a phase shift \( \phi \) is connected between nodes r and q. The phasor relationship of the voltages at various points is shown in Figure 1.26b.
Problem 1

Given the network arrangement shown in Figure 1.27, calculate \( Q \) and \( \phi \) as indicated.

![Network arrangement for problem 1](image)

Problem 2

Given a network and initial condition as shown in Figure 1.28, what happens when \( \phi \) is changed from 0\(^\circ\) to 20\(^\circ\)?

![Network and system arrangement for problem 2](image)

References

A mathematical model describing the operations of a power system contains both differential and algebraic equations. A power system, like any other dynamic system, is normally subjected to continuous perturbations. For convenience, we can assume that at a given operating level the system is at rest, i.e., an equilibrium exists. This implies that we can find a steady-state solution to the equations describing the operation of the power system. Also, in order to be viable, the system states at the equilibrium must be within acceptable ranges.

**Stability of Dynamic Systems**

Consider a dynamic system represented by a vector differential equation of the general form

\[ \dot{x} = f(x, u, t) \]

(2.1)

where \( x \) is an \( n \)-vector describing the state of the system, and \( u \) is the input vector.

If \( t \) does not appear explicitly on the right hand side of (2.1) it assumes a simpler form

\[ \dot{x} = f(x, u) \]

(2.2)

The set of nonlinear differential equations describing the dynamics of a power system is in general, at least theoretically, reducible to this form.

A particular point \( x^* \) is an equilibrium point if the system’s state at \( t_0 \) is \( x^* \) and \( x(t) = x^* \) for all \( t \geq t_0 \) in the absence of inputs or disturbances. This means \( f(x^*, 0) = 0 \) for all \( t \geq t_0 \).

The origin can be transferred to the equilibrium point by a change of variables

\[ x = x^* + x_1 \]

For this reason it can be assumed that at the equilibrium point \( x_1 = 0 \), which, after dropping the subscript, can be written as

\[ x = 0 \]

(2.3)

Stability deals with the following questions: If at time \( t_0 \) the state is perturbed from its equilibrium point, does it return to it, remain close to it, or diverge from it? Similar questions could be raised if system inputs and disturbances are allowed.

**Stability definitions**

1. The origin \( x = 0 \) is a stable equilibrium point if for a specified positive number \( \varepsilon \), no matter how small, it is possible to choose another number \( \delta(\varepsilon) \) such that \( \|x_0\| < \delta \) implies \( \|x(t)\| < \varepsilon \) for all \( t \geq t_0 \).

In the contrary case the origin is unstable.

This definition of stability is sometimes called stability in the sense of Liapunov.

2. The origin is an asymptotically stable equilibrium point if

   (a) it is stable and
   (b) \( \lim_{t \to \infty} \|x(t)\| = 0 \)
If \( \lim_{t \to \infty} \|x(t)\| = 0 \) occurs for all \( x_0 \), the origin is said to be asymptotically stable in the large or globally asymptotically stable. If, however, this requires \( x_0 \) to be sufficiently small, the origin is asymptotically stable in the small.

The above definitions of stability of dynamic systems are directly applicable to power systems.

**Power system stability definition**

A power system at a given operating state is stable if following a given disturbance, or a set of disturbances, the system state stays within specified bounds and the system reaches a new stable equilibrium state within a specified period of time.

**Comments**

Stability as defined above encompasses all types of stability likely to be encountered in power systems. Examples are: synchronous stability, shaft torsional stability, voltage stability, control loop stability, etc. It is not necessary to separately define angle (synchronous) or voltage, or any other stability for that matter. The reason is that frequently they do not manifest themselves separately, i.e., they may be interrelated. If the stability in question is known to involve only one aspect exclusively, the same definition can be interpreted to address that aspect only. For example, if the stability in question involves voltage only, the above definition would be restated as follows:

A power system at a given operating state and subject to a given disturbance is voltage stable if voltages near loads approach post-disturbance equilibrium values.

The other reason for not defining the different types of stability separately is that if the system is to be considered stable it must be stable in every sense. That is, a power system which possesses synchronous stability but not voltage stability is not acceptable. However, for practical reasons it is convenient to know the mode of instability when the system becomes unstable so that corrective measures can be planned and implemented. Two main classes of stability of primary importance in power system dynamics are:

**Small-disturbance stability**

If the magnitude of the disturbance is sufficiently small so that the system response in the initial stage is essentially linear, the stability may be classified as small-disturbance stability (or small-signal stability, or stability in the small). As will be seen later, small-disturbance stability is assured if the eigenvalues of the appropriate dynamic model, linearized about the equilibrium point, have negative real parts. If there is an eigenvalue with positive real part the system is unstable. Complex eigenvalues occur in conjugate pairs. They signify oscillations. With negative real parts oscillations damp out.

Note that when a linearized model predicts instability (one or more eigenvalues with positive real part) it does not necessarily follow that the oscillation amplitude following a disturbance will increase indefinitely. As the oscillation amplitude increases beyond a certain point, system nonlinearities and equipment limits may play a dominant role and a limit cycle may be reached. Then the true system response can only be obtained through a solution of the complete nonlinear model. Also note that in certain small-disturbance situations equipment limit may be encountered so that linearization may not be permissible, and the stability should not be classified as such.
Although negative real parts of eigenvalues of the linearized system provide sufficient conditions for small-disturbance stability, in some situations other simpler criteria may be applicable. Information on small-disturbance stability can also be obtained from a solution of the original non-linear equations using a small but finite disturbance.

**Large-disturbance stability**

Large-disturbance stability is assured if the system state at the end of the (final) disturbance lies within the region of attraction of the stable equilibrium state of the post-disturbance system. An assessment of large-disturbance stability would generally require numerical simulation. The importance of the concept of the region of attraction is that, in some situations it can be determined a priori that the disturbed state is within the region of attraction of the stable post-disturbance equilibrium, thereby obviating the need for numerical simulation.

We will offer some general comments on the terms and definitions found in the power system literature later on.

**Linear and Non-Linear Systems**

A system is defined as linear in terms of the system excitation and response. In general, a necessary condition for a linear system can be determined in terms of an excitation \( x(t) \) and a response \( y(t) \). Suppose that the system at rest is subjected to an excitation \( x_1(t) \) and the result is a response \( y_1(t) \). Also suppose that when subjected to an excitation \( x_2(t) \) the result is a corresponding response \( y_2(t) \). For a linear system it is necessary that the excitation \( x_1(t) + x_2(t) \) results in a response \( y_1(t) + y_2(t) \). This is usually called the principle of superposition. Furthermore, it is necessary that the magnitude scale factor is preserved in a linear system. Again, consider a system with an input \( x \) which results in an output of \( y \). Then it is necessary that the response of a linear system to a constant multiple \( \beta \) of an input \( x \) is equal to the response to the input multiplied by the same constant so that the output is equal to \( \beta y \). This is called the property of homogeneity. A system is linear if and only if the properties of superposition and homogeneity are satisfied.

A system characterized by the relation \( y = x^2 \) is not linear since the superposition property is not satisfied. A system which is represented by the relation \( y = mx + b \) is not linear, since it does not satisfy the homogeneity property. However, both systems can be considered linear about an operating point \( x_o, y_o \) for small changes \( \Delta x \) and \( \Delta y \).

**Linearization of non-linear systems**

A non-linear system can be linearized about an operating point, assuming small changes in variables. Consider a general non-linear element with an excitation variable \( x(t) \) and response variable \( y(t) \), the relationship of the two variables being given as

\[
y(t) = f(x(t))
\]  

(2.4)

The relationship might be shown graphically as in Figure 2.1. The normal operating point is designated by \( x_o, y_o \). Since the curve (function) is continuous over the range of interest, a Taylor series expansion about the operating point may be utilized. Then we have
STABILITY BASICS

Fig. 2.1 A graphical representation of a non-linear element.

\[ y = f(x) = f(x_o) + \frac{f'(x_o)}{1!} \Delta x + \frac{f''(x_o)}{2!} (\Delta x)^2 + \cdots \]  \hspace{1cm} (2.5)

where

\[ \Delta x = x - x_o \]

For small \( \Delta x \) higher order terms can be neglected. Therefore

\[ y = f(x_o) + f'(x_o) \Delta x \]

which yields

\[ \Delta y = f'(x_o) \Delta x \]  \hspace{1cm} (2.6)

where

\[ \Delta y = y - f(x_o) = y - y_o \]

and \( f'(x_o) = \frac{df}{dx} \bigg|_{x_o} \) is the slope at the operating point.

The same technique can be extended to systems of several variables, represented by a vector equation of the form

\[ \mathbf{y} = \mathbf{f}(\mathbf{x}) \]  \hspace{1cm} (2.7)

where \( \mathbf{y} \) is an \( n \)-vector of the dependent variables \( y_1, y_2, \cdots y_n \) and \( \mathbf{f}() \) is the \( n \)-vector function of the excitation variables \( x_1, x_2, \cdots x_m \).

By employing Taylor series expansion about the operating point \( x_{o1}, x_{o2}, \cdots x_{on} \), and neglecting higher order terms, we obtain the linearized system as

\[
\begin{bmatrix}
\Delta y_1 \\
\Delta y_2 \\
\vdots \\
\Delta y_n
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_m} \\
\frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_m}
\end{bmatrix}
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\vdots \\
\Delta x_m
\end{bmatrix}
\]  \hspace{1cm} (2.8)

which can be written as
\[ \Delta y = J \Delta x \]  

(2.9)

\( J \) is called the Jacobian of the system.

**Solution of linear differential equations with constant coefficients**

Linear time-invariant dynamic systems are represented by linear differential equations with constant coefficients. Any \( n \)th order linear differential equation can be written in the following form

\[ a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = F(t) \]  

(2.10)

\( F(t) \) is called the forcing function. The solution of equation (2.10) consists of two parts

\[ x = x_H + x_p \]  

(2.11)

The homogeneous or complementary solution, \( x_H \), is found by replacing \( F(t) \) by zero and solving the resulting homogeneous equation

\[ a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0 \]  

(2.12)

**Solution of equation (2.12)**

Assume that the solutions are of the form \( x = e^{\lambda t} \), where \( \lambda \) is a constant to be determined. Substituting the assumed solution into equation (2.12)

\[ (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) e^{\lambda t} = 0 \]

For this equation to be satisfied for all values of \( t \),

\[ a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0 \]  

(2.13)

Equation (2.13) is called the characteristic equation and can be written down directly from equation (2.12). Equation (2.13) has \( n \) roots. Denoting these roots by \( \lambda_1, \lambda_2, \cdots, \lambda_n \), the corresponding solutions to equation (2.12) are

\[ x_1 = e^{\lambda_1 t}, \quad x_2 = e^{\lambda_2 t}, \quad \cdots, \quad x_n = e^{\lambda_n t} \]

If these \( n \) solutions are linearly independent, the most general solution to the homogeneous differential equation can be written as

\[ x_H = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + \cdots + K_n e^{\lambda_n t} \]  

(2.14)

Since the coefficients in equation (2.12) are real, complex roots must occur in complex conjugate pairs. If one root is \( \lambda_1 = \alpha + j\beta \) where \( \alpha \) and \( \beta \) are real, another root must be \( \lambda_2 = \alpha - j\beta \). Then

\[
K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} = e^{\alpha t} \left( K_1 e^{j\beta t} + K_2 e^{-j\beta t} \right) \\
= e^{\alpha t} \left[ (K_1 + K_2) \cos \beta t + j(K_1 - K_2) \sin \beta t \right]
\]
For a real system, the \( a_i \)’s are real numbers, and \( x_{ni} \) is a real function of time. This means that \( K_1 \) and \( K_2 \) must be complex conjugates. Therefore, the above can be written as

\[
K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} = K e^{at} \cos(\beta t + \phi)
\]  

(2.15)

Thus each pair of complex conjugate roots give rise to an oscillatory term in the system response, the oscillation frequency being determined by the imaginary part of the roots.

The particular integral solution, \( x_p \), depends upon the input, and is also called the forced response. For a stable system, the forced solution is identical with the steady-state solution. The magnitudes of the terms in the forced solution depend upon both the input and the system parameters.

**Evaluation of the arbitrary constants**

The arbitrary constants in the complementary solution are evaluated from a knowledge of the boundary conditions or initial conditions. In most system problems, the values at \( t = 0^+ \) of \( x(t) \) and the first \( n-1 \) derivatives are used to evaluate the arbitrary constants in the solution to an \( n \)th order differential equation. The quantities are normally found by examining the stored energy in the system at \( t = 0 \). It is important to emphasize that the arbitrary constants depend upon the forcing function used and cannot be evaluated until after \( x_p \) has been found.

**Problem**

Find the complete solution for the current in the circuit shown in Figure 2.2 following closing of the switch at \( t = 0 \). Assume that the capacitor is initially uncharged.

![Fig. 2.2  An RLC circuit.](image)

**Criterion for stability of a linear system**

The form of the complementary solution depends only upon the system and not upon the input. The characteristic equation depends only upon the parameters of the system, and the roots of the characteristic equation determine the kind of terms appearing in the complementary solution. The complementary solution represents the natural behavior of the system, when it is left unexcited. For this reason, the complementary solution is also called the free or unforced response.

If the free response of a system increases without limit as \( t \) approaches infinity, the system is said to be unstable. This is the case if the characteristic equation has a root with positive real part, since the complementary solution then contains a term which increases exponentially with \( t \).
Roots with negative real parts, on the other hand, lead to terms that become zero as $t$ approaches infinity. Purely imaginary roots, if they are simple, lead to sinusoidal terms of constant amplitude in the complementary solution. This case of constant amplitude oscillation in the complementary solution, which is characteristic, for example, of $LC$ circuits, is usually considered a stable and not an unstable response. Repeated imaginary roots lead to terms of the form $t^n \cos(\omega t + \phi)$, which is an unstable response. If the roots of the characteristic equation are plotted in a complex plane, the following statement can be made:

For a stable system, none of the roots can lie in the right half-plane, and any roots on the imaginary axis must be simple. If all the roots of the characteristic equation lie in the left half-plane, the system is asymptotically stable.

**Concepts of damping ratio and natural frequency**

Consider the following second-order system

$$a_2 \ddot{x} + a_1 \dot{x} + a_0 x = F(t)$$

(2.16)

The characteristic equation of this system is

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

(2.17)

Equation (2.17) can be written as

$$\lambda^2 + 2\xi \omega_n \lambda + \omega_n^2 = 0$$

(2.18)

in terms of the dimensionless damping ratio $\xi$ and the natural frequency $\omega_n$ of the system. In this case

$$\omega_n = \sqrt{a_o / a_2}, \text{ and } \xi = a_1 / (2\sqrt{a_o a_2})$$

The roots of the characteristic equation are

$$\lambda_1, \lambda_2 = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

(2.19)

When $\xi > 1$, the roots are real, and when $\xi < 1$, the roots are complex conjugates. When $\xi = 1$, the roots are repeated and real and the condition is called critical damping.

When $\xi < 1$, the response is underdamped and

$$\lambda_1, \lambda_2 = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2}$$

(2.20)

The complementary solution corresponding to this case, from (2.15),

$$x_{ch} = K e^{-\xi \omega_n t} \cos(\sqrt{1 - \xi^2} \omega_n t + \phi)$$

(2.21)

As $\xi$ varies with $\omega_n$ constant, the roots follow a circular locus as shown in figure 2.3. The transient response is increasingly oscillatory as the roots approach the imaginary axis when $\xi$ approaches zero.
Fig. 2.3 Locus of roots as $\xi$ varies with $\omega_n$ constant.

Transient response for a unit step input ($F(t) = 1$) for various values of the damping ratio $\xi$ is shown in Figure 2.4. It can be seen that for $\xi = 2.0$ the system is overdamped.

Fig. 2.4 Transient response of a second order system for a step input.

Although critical damping, $\xi = 1.0$, produces a response with no overshoot, it is evident that a system with a lower value of $\xi$, for instance, $0.6 < \xi < 1.0$, approaches the steady state more quickly (an optimum value for $\xi$ is 0.707). Furthermore, the time for the response to come and remain within the vicinity of the steady state (say, within $\pm$ 5 percent) is shorter. However, further reduction in $\xi$ to 0.4 and lower produces a definitely oscillatory response that can be quite objectionable.
The State Variables of a Dynamic System

The time-domain analysis of dynamic systems utilizes the concept of the state of a system. The state of a system is a set of numbers or variables (state variables), such that a knowledge of these along with the input functions (control forces) and the equations describing the dynamics of the system, is sufficient to completely determine the future behavior (state and output) of the system.

Consider the system shown in Figure 2.5, where \( y_1(t) \) and \( y_2(t) \) are the output signals and \( u_1(t) \) and \( u_2(t) \) are the input signals. A set of state variables \( x_1, x_2, \ldots, x_n \) for the system shown in Figure 2.5 is a set such that a knowledge of the initial values of the state variables \( x_1(t_o), x_2(t_o), \ldots, x_n(t_o) \) at the initial time \( t_o \), and of the input signals \( u_1(t) \) and \( u_2(t) \) for \( t \geq t_o \), suffices to determine the future values of the outputs and state variables.

![Fig. 2.5 Block diagram of a dynamic system.](image)

The concept of a state variable description of a system can be illustrated in terms of the RLC circuit shown in Figure 2.6. The number of state variables chosen to represent a system should not exceed the minimum number required so as to avoid redundancy. The state of the system in Figure 2.6 may be described in terms of a set of state variables \( x_1, x_2 \) where \( x_1 \) is the capacitor voltage \( v_c(t) \) and \( x_2 \) is the current \( i(t) \) flowing through the circuit. (For a passive RLC network, the number of state variables required is equal to the number of independent energy storage elements.)

![Fig. 2.6 An RLC circuit.](image)

Applying Kirchhoff’s voltage law to the circuit of Figure 2.6,

\[
Ri + L \frac{di}{dt} + v_c = E(t) \tag{2.22}
\]

Also
\[ i = C \frac{dv_c}{dt} \]  \hspace{1cm} (2.23)

The output, \( v_L \), is given by

\[ v_L = L \frac{di}{dt} \]  \hspace{1cm} (2.24)

We may rewrite (2.22) and (2.23) in terms of the state variables \( x_1 \) \((= v_c(t))\) and \( x_2 \) \((= i(t))\) as follows:

\[ \frac{dx_1}{dt} = \frac{1}{C} x_2 \]  \hspace{1cm} (2.25)

\[ \frac{dx_2}{dt} = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} E(t) = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u(t) \]  \hspace{1cm} (2.26)

The output signal is then

\[ v_L = y(t) = L \frac{dx_2}{dt} = -x_1 - R x_2 + u(t) \]  \hspace{1cm} (2.27)

Utilizing equations (2.25) and (2.26) and the initial conditions of the network represented by \((x_1(t_0), x_2(t_0))\), we may determine the system's future behavior and its output.

Consider the spring-mass-damper system shown in Figure 2.7.

\[ \begin{align*}
\text{Fig. 2.7} & \quad \text{A spring-mass-damper system.} \\
\text{The differential equation describing the behavior of the system can be written as} \\
M \frac{d^2 y}{dt^2} + f \frac{dy}{dt} + K y &= u(t) \\
\text{A set of state variables sufficient to describe this system is the position and velocity of the mass.} \\
\text{Therefore, we can define a set of state variables as} \ (x_1, x_2) \ \text{where} \\
x_1(t) &= y(t), \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt} \\
\textbf{Problem} \\
\text{Write the state equations for the system shown in Figure 2.7.}
\end{align*} \]
The state variables which describe a system are not a unique set. Several alternative sets of state variables may be chosen. A good choice is a set of state variables which can be readily measured.

**The state vector differential equation**

A dynamic system can be described by a set of first-order differential equations written in terms of the state variables. The first-order differential equations of a linear system may be written in a general form as

\[
\begin{align*}
\dot{x}_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + b_{11} u_1 + \cdots + b_{1m} u_m \\
\dot{x}_2 &= a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n + b_{21} u_1 + \cdots + b_{2m} u_m \\
&\vdots \\
\dot{x}_n &= a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n + b_{n1} u_1 + \cdots + b_{nm} u_m
\end{align*}
\]

(2.29)

where \( \dot{x} = dx/dt \), and some of the coefficients \( a_{ij} \) and \( b_{ij} \) may be zero.

This set of simultaneous differential equations may be written in a matrix form as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} +
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nm}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_m
\end{bmatrix}
\]

or

\[
\dot{x} = Ax + Bu
\]

(2.30)

\( x \) is the vector of the state variables and \( u \) is the vector of input signals. The matrix \( A \) is an \( n \times n \) matrix and \( B \) is an \( n \times m \) matrix. The vector matrix differential equation relates the rate of change of the state of the system to the state of the system and the input signals. The matrix \( A \) is also known as the coefficient matrix or the system matrix. In general, the outputs of a linear system may be related to the state variables and the input signals by the vector matrix equation

\[
y = Cx + Du
\]

(2.31)

where \( y \) is the set of output signals expressed in a column vector form.

The \( n \)th order linear differential equation

\[
\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = u(t)
\]

can be reduced to the form of \( n \) first-order state equations as follows:

Defining the state variables as

\[
x_1 = x, \quad x_2 = \frac{dx}{dt}, \quad x_3 = \frac{d^2 x}{dt^2}, \quad \cdots, \quad x_n = \frac{d^{n-1} x}{dt^{n-1}}
\]

the \( n - 1 \) first-order differential equations are

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \cdots, \quad \dot{x}_{n-1} = x_n
\]
The $n$th equation is
\[ \frac{d^n x}{dt^n} = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u(t) \]
so that
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
& & & & & \\
& & & & & \\
-a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \vdots \\ \ddots \\ \ddots \\ \ddots \\ \ddots \end{bmatrix} + u(t) = A x + B u(t)
\]

The output is
\[ x(t) = x_i(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x(t) = C x(t) \]

The solution of the state vector differential equation (2.30) may be obtained in a manner similar to that used for solving a first-order differential equation.

As in the case of an $n$th order differential equation, the solution to the linear unforced system
\[ \dot{x} = A x \tag{2.32} \]
will describe the character of the response of the system. As before, the solution to equation (2.32) is of exponential form. If we let $x = e^{\lambda t}$ and substitute into (2.32), we have
\[ \lambda e^{\lambda t} = A e^{\lambda t} \]
or
\[ \lambda e = A e \tag{2.33} \]

Equation (2.33) may be written as
\[ [\lambda I - A] e = 0 \tag{2.34} \]
where $I$ is the identity matrix and $0$ equals the null vector. Equation (2.34) has a nontrivial solution if, and only if, the determinant of the matrix $[\lambda I - A]$ vanishes, i.e., if
\[ |\lambda I - A| = 0 \tag{2.35} \]

The equation resulting from the evaluation of this determinant is the characteristic equation. It is a polynomial in $\lambda$. For an $n$th order system the equation will have $n$ roots, called the characteristic roots or the eigenvalues of the system. The stability of the system may be readily ascertained from an examination of the characteristic roots.

**Problems**

1. Show that the characteristic equation of the spring-mass-damper system shown in Figure 2.7 is the same whether derived from equation (2.28) or from the state equations describing the system.

2. Convert the following set of first-order equations (state equations) into a second-order differential equation and show that the characteristic equations in both representations are the same.
The choice of state variables describing a system is not unique. For example, another set of state variables $z$ could be used to describe the system given by (2.30) where $z$ is related to $x$ by the relationship

$$x = Tz$$

(2.36)

where $T$ is the transformation matrix. In order for the transformation to be reversible $T$ must be non-singular.

Substituting (2.36) into (2.30), we have

$$\dot{z} = T^{-1} ATz + T^{-1} Bu$$

(2.37)

Equation (2.37) describes the same system as (2.30) and the characteristic roots are not altered in the transformation (verify this).

If the system is unstable (i.e. some of the characteristic roots of $A$ have positive real parts) or has unacceptable transient response, the system can be stabilized and the performance improved by utilizing feedback of the state variables. If state variable feedback is used so that $u = -Hx$, where $H$ is the feedback matrix, equation (2.30) reduces to

$$\dot{x} = [A - BH]x = A_1 x$$

(2.38)

The elements of the matrix $H$ can be so chosen that the modified system given by equation (2.38) will have the desired performance.

**Solution of the equation** $\dot{x} = Ax$

The solution is of the exponential form given by

$$x = e^{\lambda t}$$

(2.39)

As noted earlier, $\lambda$ is obtained by solving equation (2.35), which is a polynomial in $\lambda$. For an $n$th order system the equation has $n$ roots called the characteristic roots.

The vector $e$ is obtained from a solution of equation (2.33). The vector $e$ obtained for each $\lambda$ is called a characteristic vector or an eigenvector of the system. It may be noted that equation (2.33) is satisfied by multiplying the vector $e$ by any constant $k$. Therefore, the absolute values of the elements of the eigenvectors are not determined. Only the direction of each eigenvector is unique.

The complete solution of the equation $\dot{x} = Ax$ can therefore be written as, assuming that the roots are distinct,

$$x(t) = k_1 e_1 e^{\lambda_1 t} + k_2 e_2 e^{\lambda_2 t} + \cdots + k_n e_n e^{\lambda_n t} = \sum_{i=1}^{n} e_i e^{\lambda_i t} k_i$$

(2.40)

where $k_1$, $k_2$, etc. are arbitrary constants to be determined from given initial conditions; $e_1$, $e_2$, etc. are the eigenvectors corresponding to the eigenvalues $\lambda_1$, $\lambda_2$, etc.

Equation (2.40) can be written in matrix form as
\( x(t) = M Q(t) k \)  

where

\[
M = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}
\]  

is a matrix composed of the eigenvectors \( e_1, e_2, \ldots, e_n \) called the modal matrix,

\[
Q(t) = \begin{bmatrix} e_1^{\lambda t} & e_2^{\lambda t} & \cdots & e_n^{\lambda t} \end{bmatrix}
\]

and

\[
k = \begin{bmatrix} k_1 \\
k_2 \\
\vdots \\
k_n \end{bmatrix}
\]

If the initial value of \( x \) is given by \( x_0 \), then

\[
x_0 = M k
\]

from which

\[
k = M^{-1} x_0
\]

Therefore, the solution can be written as

\[
x(t) = M Q(t) M^{-1} x_0
\]

**Example**

We will obtain the solution of the second-order equation

\[
\frac{d^2 x}{dt^2} = -\omega^2 x
\]

by the above method.

Equation (2.44) can be written in state space form as

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

where

\[
x = x_1, \quad \dot{x} = \dot{x}_1 = x_2
\]

The eigenvalues of the system are

\[
\lambda_{1,2} = \pm j \omega
\]

We obtain the eigenvectors as follows

\[
j\omega \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix}
\]
from which

\[ j\omega e_{11} = e_{21} \]
\[ j\omega e_{21} = -\omega^2 e_{11} \]

with \( e_{11} = 1, \quad e_{21} = j\omega \)

For the second eigenvector

\[-j\omega \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix}\]

from which

\[-j\omega e_{12} = e_{22} \]
\[-j\omega e_{22} = -\omega^2 e_{12} \]

with \( e_{12} = 1, \quad e_{22} = -j\omega \)

Therefore, the modal matrix is

\[ M = \begin{bmatrix} 1 & 1 \\ j\omega & -j\omega \end{bmatrix} \]

and

\[ M^{-1} = \begin{bmatrix} 1/2 & -j/2\omega \\ 1/2 & j/2\omega \end{bmatrix} \]

If

\[ x_o = \begin{bmatrix} x_{1o} \\ 0 \end{bmatrix} \]

then the solution is, from equation (2.43),

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j\omega & -j\omega \end{bmatrix} \begin{bmatrix} e^{j\omega t} \\ e^{-j\omega t} \end{bmatrix} \begin{bmatrix} 1/2 & -j/2\omega \\ 1/2 & j/2\omega \end{bmatrix} \begin{bmatrix} x_{1o} \\ 0 \end{bmatrix} \]

\[ = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j\omega & -j\omega \end{bmatrix} \begin{bmatrix} x_{1o} e^{j\omega t} \\ x_{1o} e^{-j\omega t} \end{bmatrix} = \begin{bmatrix} x_{1o} \cos \omega t \\ -\omega x_{1o} \sin \omega t \end{bmatrix} \]

from which

\[ x_1 = x_{1o} \cos \omega t, \quad x_2 = -\omega x_{1o} \sin \omega t \]

**Problem**

Show that if the initial condition vector is

\[ x_o = \begin{bmatrix} x_{1o} \\ x_{2o} \end{bmatrix} \]

the solution is of the form

\[ x_1 = A \cos(\omega t - \phi) \]
Since the modes of a system response, i.e., whether it is oscillatory, damped or undamped, etc., are determined by the eigenvalues, the eigenvalues are often referred to as the system modes. The relative impact of a particular mode on a system variable depends on the relative magnitude of the element of the corresponding eigenvector. For this reason, the eigenvectors can be considered as describing the mode shapes.

From equation (2.43) it should be clear that the overall impact of the various modes on the system response depends on the eigenvectors as well as the initial condition. For example, if the initial perturbation is in the direction of one of the eigenvectors, say the eigenvector corresponding to the eigenvalue $\lambda_1$, i.e., if

$$x_0 = \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \\ \vdots \end{bmatrix} \alpha$$

then

$$M^{-1}x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \alpha, \text{ since } M^{-1}M = I$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_{11} \alpha e^{\lambda_1 t} \\ e_{21} \alpha e^{\lambda_1 t} \\ \vdots \end{bmatrix}$$

Therefore, all the modes except the one corresponding to $\lambda_1$, are absent.

**An alternative derivation of equation (2.43)**

If equation (2.33) is written for each eigenvalue and the corresponding eigenvector and combined, we have

$$M \Lambda = \Lambda M$$

where $M$ is the modal matrix as defined previously and $\Lambda$ is a diagonal matrix composed of the eigenvalues, i.e.,

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots)$$

From (2.46)

$$M^{-1} \Lambda M = \Lambda$$

If we now write equation $\dot{x} = \Lambda x$ in terms of another set of variables $z$ using the transformation

$$x = Mz$$

we have

$$M \dot{z} = \Lambda M z$$

from which

$$\dot{z} = M^{-1} \Lambda M z = \Lambda z$$
The variables in equation (2.49) are completely decoupled. The solution to equation (2.49) is therefore
\[ z = Q(t)z_o \] (2.50)
where the matrix \( Q(t) \) is as defined earlier; \( z_o \) is obtained from equation (2.48) as
\[ z_o = M^{-1}x_o \]
which yields
\[ x = Mz = MQ(t)M^{-1}x_o \] (2.43)
Note that by definition
\[ e^{At} = I + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \]
\[ Q(t) = \begin{bmatrix} e^{\lambda_1t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_nt} \end{bmatrix} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = e^{At} \]
\[ MQ(t)M^{-1} = M \begin{bmatrix} I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \end{bmatrix} M^{-1} = e^{At} \]
Therefore the solution (2.43) can also be written as
\[ x = e^{At}x_o \] (2.51)
Equation (2.51) also follows directly from the solution of the first-order equation by analogy. The matrix \( e^{At} \) is known as the fundamental or transition matrix, usually denoted by \( \phi(t) \).

It can be shown that the solution to the equation (2.30) is
\[ x(t) = e^{At}x_o + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \] (2.52)

**Liapunov’s Direct Method of Stability Analysis**

We will now review the so-called Liapunov’s direct method of stability analysis.

A single valued function \( V(x) \) is said to be positive (negative) definite in some neighborhood of the origin, if \( V(0) = 0 \) and \( V(x) > 0 \) (< 0) for \( x \neq 0 \). If \( V(0) = 0 \) and \( V(x) \geq 0 \) (\( \leq 0 \)), the function is said to be positive (negative) semidefinite.

**Stability and instability theorems**

Consider the autonomous system
\[ \dot{x} = f(x) \] (2.53)
The origin is assumed to be an equilibrium point.
**Theorem 1**: Suppose there exists a positive definite function $V(x)$ which is continuous together with its first partial derivatives in some neighborhood of the origin and whose total derivative $\dot{V}(x)$ along every trajectory of (2.53) is negative semidefinite, then the origin is stable.

The function $V(x)$ is called a Liapunov function.

**Theorem 2**: If $\dot{V}(x)$ is negative definite, then the stability is asymptotic.

An extension of the above theorem is:

**Theorem 3**: If $\dot{V}(x)$ is negative semidefinite but not identically zero along any trajectory, then the origin is asymptotically stable.

**Theorem 4**: The origin is a globally asymptotically stable equilibrium point if a Liapunov function $V(x)$ can be found such that (i) $V(x) > 0$ for all $x \neq 0$ with $V(0) = 0$, (ii) $\dot{V}(x) < 0$ for all $x \neq 0$, and (iii) $V(x) \to \infty$ as $\|x\| \to \infty$.

**Theorem 5**: Let $V(x)$ with $V(0) = 0$ have continuous first partial derivatives in some neighborhood of the origin. Let $\dot{V}(x)$ be positive definite and let $V(x)$ be able to assume positive values arbitrarily near the origin. Then the origin is unstable.

The proof of the above theorems are based on the fact that $V(x)$ being positive definite $V = C = \text{constant}$ represents a one-parameter family of closed surfaces surrounding the origin in the neighborhood of the origin. For $\dot{V}(x)$ negative definite any initial state sufficiently close to the origin must eventually approach the origin.

**Illustration of positive definite functions and their closedness**

Of the three functions

$$V_1 = x_1^2 + x_2^2 + x_3^2, \quad V_2 = (x_1 + x_2)^2 + x_3^2, \quad \text{and} \quad V_3 = x_1^2 - x_2^2 + x_3^2$$

the function $V_1$ is positive definite, the function $V_2$ is positive semidefinite and the function $V_3$ is indefinite.

For the above positive definite function $V_1$, $V = C$ surfaces are closed for any $C$. In general, however, the surfaces $V = C$ are closed only if $C$ is sufficiently small. An example is provided by the function

$$V = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

This defines a family of closed curves $V = C$ only if $C \leq 1$. Consequently, this can serve as a Liapunov function only for an investigation of stability in which the disturbances are limited by the condition

$$\frac{x_{10}^2}{1 + x_{10}^2} + x_{20}^2 < 1$$

Another example is given by the function
\[ V = \int_{0}^{x_1} \phi(x)dx + x_2^2 \]
where \( \phi(x) \) satisfies the conditions
\[ \phi(0) = 0, \quad x_1 \phi(x_1) > 0 \text{ for } x_1 \neq 0 \]

The curves \( \int_{0}^{x_1} \phi(x)dx + x_2^2 = C \) naturally must be closed if \( C \) is sufficiently small. But if \( \int_{0}^{x_1} \phi(x)dx \) tends to a limit \( a \) as \( x_1 \to \infty \), then the curves \( V = C \) will be closed only for \( C < a \). This emphasizes the need to verify whether the curves \( V = C \) are closed whenever \( C \) is not sufficiently small.

Closure of the curves is assured if, in addition to the above, \( V \) approaches infinity as \( \|x\| \to \infty \). In this case \( V \) is said to be unbounded.

The simplest and a very useful positive definite function is a quadratic form which may be written in matrix form as
\[ V(x) = x'Px \quad (2.54) \]
where \( P \) is a real symmetric matrix.

The necessary and sufficient conditions in order that \( V(x) \) be positive definite are that the successive principal minors of the matrix \( P \) be positive, i.e.,
\[ p_{11} > 0, \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{vmatrix} > 0 \cdots \quad (2.55) \]

This test is, however, unsuitable for large matrices due to the inconvenience of evaluating determinants of large order. A much better method is to use the fact that \( P \) is positive if and only if it can be expressed as the product of two matrices, \( LL' \), where \( L \) is a real lower triangular nonsingular matrix. The test for nonsingularity can be made very easily as the determinant of a triangular matrix is simply the product of the diagonal elements.

**Stability determined by linear approximation**

Considering again the equation \( \dot{x} = f(x) \), in the majority of problems \( f(x) \) can be expanded into a convergent power series in a sufficiently small region about the origin. Then (2.53) can be written as
\[ \dot{x} = Ax + F(x) \quad (2.56) \]
where \( A \) is a constant nonsingular matrix and the vector function \( F(x) \) does not contain terms of order less than 2.

A sufficient condition for the origin of the nonlinear system (2.56) to be asymptotically stable is that the eigenvalues of the matrix \( A \) all have negative real parts. If there is an eigenvalue with positive real part the origin is unstable.

The critical cases are those for which several of the eigenvalues of the matrix \( A \) of the linear terms are zero or pure imaginary in pairs, while the remaining roots have negative real parts. In
all these cases the stability cannot be determined by investigating only the linear terms. In critical cases the stability is determined by the form of the nonlinearities and then it becomes necessary to consider equation (2.53) in its original form. However, for our purposes it will not be necessary to deal with these special cases.

The above theorem of Liapunov is important because it puts on a strong mathematical basis the engineer’s linearization technique in studying stability under small disturbances.

**Extent of asymptotic stability**

Asymptotic stability is a local concept, and merely to have established asymptotic stability does not necessarily mean that the system will operate satisfactorily. Some knowledge of the size of the region of asymptotic stability is always desirable.

There are several ways of determining the extent of asymptotic stability. One which is useful in power system problems is as follows. The Liapunov function $V$ is formed such that $\dot{V}$ is at least negative semidefinite in the whole state space, and $V$ is positive definite only in a finite region around the origin. If the largest $V = \text{const.}$ closed surface is found, the region within this surface is the region of stability.

**Construction of Liapunov functions**

The Liapunov function $V(x)$ may be thought of as a generalized energy. In many problems the energy function can serve as a Liapunov function. In cases where a system model is described mathematically, it may not be clear what “energy” means. The conditions which $V(x)$ must satisfy in order to be a Liapunov function are therefore based on mathematical rather than physical consideration.

Although Liapunov’s theorems give sufficient conditions for stability or asymptotic stability they do not give any indication of how a Liapunov function can be constructed in the general case. Also, for a particular system there is not a unique Liapunov function. The fact that a particular Liapunov function ensures stability in a certain region around the equilibrium point, does not necessarily mean that the motions are unstable outside this region. On the other hand for a stable or asymptotically stable equilibrium state a Liapunov function with the required properties always exists.

A lot of work has been done in developing systematic methods of construction of Liapunov functions. None of the methods devised so far is really effective in generating Liapunov functions for high order systems with general nonlinearities.

A practical power system is described by nonlinear differential equations of very high order, and construction of Liapunov functions other than those suitable for studying stability in the small, is impractical. Claims have been made in the power system literature of successful use of Liapunov method (also known as energy function method, direct method) in stability analysis of large power systems. The theory of Liapunov’s direct method has been discussed in some detail here, so the readers can verify for themselves that all these works, including [10], are mathematically compromised.

For a linear time-invariant system given by $\dot{x} = Ax$, the scalar function $x'Px$ can be chosen as a Liapunov function. If $V(x) = x'Px$, then $\dot{V}(x) = x'(A'P + PA)x = -x'Qx$, where

$$A'P + PA = -Q$$  \hspace{1cm} (2.57)
If \( Q \) is chosen to be positive definite, the necessary and sufficient conditions for the linear system to be asymptotically stable is that \( P \) is positive definite. Equation (2.57) has a unique solution for \( P \) if \( \lambda_i + \lambda_j \neq 0 \) for all \( i, j \), where \( \lambda_i \) are the eigenvalues of the matrix \( A \). Using this approach, the conditions for a positive definite \( P \) matrix, in terms of the signs of the principal minors, lead to conditions for stability in terms of the \( a_{ij} \) coefficients. The stability conditions which are obtained in this manner are of the same type as obtained from Routh’s criterion (see Appendix B).

**Comments on Certain Terms in Common Use**

**Dynamic stability**

Much confusion has existed regarding the use of the expression “dynamic stability.” Originally the term was intended to address small-disturbance stability in the presence of excitation (and possibly governor) control. However, over time the term has been used to address other phenomena (including large-disturbance (transient) stability in some parts of the world). This led to the discouragement of further use of the term [12]. There is, however, a more fundamental reason to exclude the term. The question of stability or instability arises only in dynamic systems, i.e., in systems where changes can and do occur. Therefore, the word “dynamic” in the above expression is redundant. If used, it can justifiably refer to any and all types of stability or instability and, in the absence of a universally agreed upon usage, this is precisely what happened in the past.

**Transient/short-term/long-term stability**

The same comment as applied to “dynamic stability” also applies to the term “transient stability.” The question of stability arises during the transients following a disturbance or disturbances. If there were no transients there would be no instability. Therefore, the word “transient” is redundant, and if used the term can logically refer to both large- and small-disturbance stability. However, unlike “dynamic stability,” this term never caused any confusion as to its intended meaning. This is probably because it was one of the original terms defined in power system stability and it has been ingrained in the engineers’ mind. The current thinking seems to favor the expression “large-disturbance stability” so as to be compatible with the terminology in related fields [11].

When transient stability was originally defined, large-disturbance stability could be assessed by observing the system response for a brief period immediately following the disturbance. If the system was stable during the first swing it could be concluded that the system would be stable in the long run. As power systems expanded and systems were interconnected it became necessary to simulate several swings before conclusion about stability or instability could be drawn. Depending on the disturbance and the extent of system disruption and control actions that followed the disturbance, sometimes it became necessary to extend the simulation to many seconds or even minutes of real time. To cover these various lengths of simulations, several new terms were coined. These are: “short-term stability,” “mid-term stability,” and “long-term stability.” One purpose of this classification was to identify the modeling and solution requirement for simulations covering the various time periods [12].

We note that to be considered stable a power system must be stable in the long term. A power system possessing transient or short-term stability as observed from simulation is not of much use if it cannot be operated stably on a long term basis. Therefore the use of terms like “short-
term stability,” etc. seems odd. However, it would seem logical to use the terms “transient dynamics,” “short-term dynamics,” etc. to refer to the dynamics covering these periods. We can then say that in a stable system all the dynamics, from transient to long term, are stable. If any of these is unstable the system is unstable. In most cases experience will dictate that not all these dynamics need to be analyzed to ensure stability.

**Steady-state stability**

Originally the expression was used to refer to small-disturbance stability under fixed or manual control. Later it was retained in order to distinguish from stability under automatic control which was termed “dynamic stability” [12]. The expression represents a contradiction in terms. If something is in a steady state, it is stable by definition. However, there is no harm if it is understood that it refers to small-disturbance stability.

**Natural or inherent stability/conditional stability**

This classification of stability was mentioned in [12]. Conditional stability was stated as stability under the action of automatic control as opposed to natural or inherent stability where no control action is required. The comment was made that since automatic controls are inherent parts of the system, the concept of natural or inherent stability is redundant. We think that, if desired, the terms may still be retained, although defined somewhat differently reflecting the current concepts in emergency controls in power systems. Many power systems are designed to survive certain disturbances through such control actions as generator tripping, load shedding, etc., much of which is initiated by the system operator. When stability is maintained through such operator action(s), the stability may be termed conditional stability (although in a strict sense this is not stability). If no such control action is required, the system can be considered to possess inherent stability.

**Stability limits**

Two types of stability limits are in general use — small-disturbance (steady-state) limit and large-disturbance (transient) limit. In a given power system the large-disturbance stability limit cannot exceed the small-disturbance stability limit, although in some cases the former can closely approach the latter. This may not always be apparent from simulation results, where large-disturbance stability simulation results covering a short period of time may suggest a higher limit than the small-disturbance limit as determined from a linearized analysis. Herein lies the importance of small-disturbance limit.

A power system cannot be operated above its small-disturbance stability limit although, depending on the operating criteria, it may be permissible to operate the system above the large-disturbance limit. This means, whenever there is a disturbance of sufficiently large magnitude, there will be system disruption and shut down. Depending on the probability of occurrence of such disturbances this may be an acceptable operating mode. Then the small-disturbance limit assumes special importance. This also points to the justification of combining the two terms and simply call it stability limit.

**Classification of instability**

There has been a trend in classifying angle and voltage instability separately. The problem with separately classifying angle and voltage instability is that frequently these two go together and then another term would be needed to identify such cases, although when angle or voltage instability manifests itself distinctly, one without involving the other, such separate classification
can be helpful. A similar comment applies to unidirectional and oscillatory instabilities. When
the instability involves generators, unidirectional and oscillatory instabilities are caused by lack
of sufficient synchronizing torque and damping torque, respectively. However, instability due to
the lack of both synchronizing and damping torque is not uncommon.

Lack of damping or negative damping in the generator rotor oscillations is generally contributed
by the generator excitation control. It has been shown that oscillatory instability can also be
caused by the interaction of excitation control and load dynamics without the participation of the
generator angle [13]. Although it would be tempting to classify such instability under voltage
instability, it is actually a control loop instability [14]. It is shown in Chapter 10 that true voltage
instability is caused by unfavorable load characteristics.

Series capacitors and certain HVDC controls can also contribute to negative damping.

References

1. J.P. La Salle and S. Lefschetz, Stability by Lyapunov's Direct Method with Applications, Academic Press,
   1963.
    Birmingham, 1972.
CHAPTER 3
SYNCHRONOUS MACHINE STABILITY BASICS

The Swing Equation

Applying the laws of mechanics to the rotational motion of a synchronous machine

\[ I \frac{d^2 \theta}{dt^2} = T_a \]  \hspace{1cm} (3.1)

where \( I \) is the moment of inertia of the rotor system, \( \theta \) is the mechanical angle of the rotor in radians with respect to a fixed reference, and \( T_a \) is the net torque acting on the machine.

The net torque \( T_a \) is the accelerating (or retarding) torque given by

\[ T_a = T_m - T_e \] \hspace{1cm} (3.2)

where \( T_m \) is the shaft mechanical torque, corrected for rotational losses, and \( T_e \) is the electromagnetic torque.

In the steady state \( T_a = 0 \). If we measure the angular position and velocity with respect to a synchronously rotating reference axis instead of with respect to a stationary axis,

\[ \delta = \theta - \omega_o t \] \hspace{1cm} (3.3)

from which

\[ \frac{d\delta}{dt} = \frac{d\theta}{dt} - \omega_o \] \hspace{1cm} (3.4)

and

\[ \frac{d^2 \delta}{dt^2} = \frac{d^2 \theta}{dt^2} \] \hspace{1cm} (3.5)

Therefore, equation (3.1) becomes

\[ I \frac{d^2 \delta}{dt^2} = T_a \] \hspace{1cm} (3.6)

Equation (3.6) can be written as

\[ I \omega_o \frac{d^2 \delta}{dt^2} = T_a \omega_o \]

or

\[ M \frac{d^2 \delta}{dt^2} = T_a \omega_o \] \hspace{1cm} (3.7)

where \( M = I \omega_o \) is the angular momentum. At normal speed \( \omega_o \), the value of \( M \) is called the inertia constant of the machine.

Equation (3.6) can also be written in terms of the stored kinetic energy at rated speed.
SYNCHRONOUS MACHINE STABILITY BASICS

\[
\frac{2}{\omega_o} \frac{1}{2} I \omega_o^2 \frac{d^2 \delta}{dt^2} = T_a \omega_o
\]

or

\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = T_a \omega_o \quad (3.8)
\]

where

\[
H = \frac{1}{2} I \omega_o^2 \quad (3.9)
\]

From (3.9)

\[
H = \frac{1}{2} M \omega_o
\]

or

\[
M = \frac{2H}{\omega_o} \quad (3.10)
\]

The mechanical angle and speed are related to the electrical quantities by the relations

\[
\delta_e = \frac{p}{2} \delta_m, \quad \omega_e = \frac{p}{2} \omega_m
\]

where \(p\) is the number of poles.

Since mechanical input and electrical output are usually expressed in terms of power, it is convenient to convert torque into power using the relation

\[
T \omega = P \quad (3.11)
\]

Therefore, equation (3.8) reduces to

\[
\frac{2H}{\omega_o} \frac{d^2 (\delta P)}{dt^2} = \frac{P_a}{2} \omega_o
\]

or

\[
\frac{2H}{\omega_e} \frac{d^2 \delta_e}{dt^2} = \frac{P_a}{\omega_e} = \frac{P_a}{\omega_{ae} + \Delta \omega_e}/\omega_{ae} = P_a \left(1 - \Delta \omega_e/\omega_{ae}\right)
\]

or

\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P_a \left(1 - \Delta \omega_e/\omega_o\right) \quad (3.12)
\]

In equation (3.12) the subscript \(e\) has been dropped, since it is understood that \(\delta\) and \(\omega\) refer to electrical quantities.

Similarly, equation (3.7) reduces to

\[
\frac{2M}{P} \frac{d^2 \delta}{dt^2} = P_a \left(1 - \Delta \omega_e/\omega_o\right) \quad (3.13)
\]
Selection of units

$\omega_{om}$ in radian/sec is calculated from the given rpm of the machine, $n$.

$$\omega_{om} = \frac{n}{60} \cdot 2\pi$$

(3.14)

In SI units $I$ will be given in kg-m$^2$, and $H$ will be in joules or watt-sec. When multiplied by $10^6$, this will be expressed in mega joules or megawatt-sec.

Equation (3.12) can be expressed in per unit by dividing both sides by the MVA base.

$$\frac{2H}{\omega_{om}MVA_b} \frac{d^2\delta}{dt^2} = \frac{P_a}{MVA_b} (1 - \Delta\omega_e / \omega_o)$$

or

$$\frac{2H_{pu}}{\omega_{om}} \frac{d^2\delta}{dt^2} = P_{a_{pu}} (1 - \Delta\omega_e / \omega_o)$$

(3.15)

where $H_{pu}$ is given in terms of MWs/MVA.

The manufacturers in the United States usually give the value of the moment of inertia in pound-feet$^2$ (called $WR^2$). The value of $H$ in MWs/MVA can then be computed in one of the following ways.

(a) Convert the value of the moment of inertia in pound-feet$^2$ to the SI unit kg-m$^2$, using the relationships:

$$1 \text{ lb} = 0.4537 \text{ kg}, \quad 1 \text{ ft} = 0.3048 \text{ m}$$

Then, if $WR^2 =$ moment of inertia in lb-ft$^2$, the inertia in SI unit is

$$WR^2 \times 0.4537 \times 0.3048^2 \quad \text{kg} \cdot \text{m}^2$$

Therefore

$$H = \frac{1}{2} \frac{WR^2 \times 0.4537 \times 0.3048^2 \times \frac{n^2}{3600}}{4\pi^2} \quad \text{Ws}$$

$$= \frac{1}{2} \frac{WR^2 \times \frac{n^2}{3600}}{4\pi^2} \times 0.4537 \times 0.3048^2 \times 10^{-6} \quad \text{MWs}$$

(3.16)

(b) Alternatively

$$H = \frac{1}{2} \frac{WR^2}{g} \times \frac{n^2}{3600} \times 4\pi^2 \quad \text{lb-ft}$$

where $g$ is the acceleration due to gravity = 32.2 ft/sec$^2$

$$= \frac{1}{2} \frac{WR^2 \times \frac{n^2}{3600}}{4\pi^2} \left( \frac{746}{g \times 550} \right) \times 10^{-6} \quad \text{MWs}$$

(3.17)

It may be observed that the quantities within the parentheses in equations (3.16) and (3.17) are equal, as they should be.
Combining all the constants in equation (3.16) or (3.17), the expression for \( H \) can be written as

\[
H = 2.31 \times 10^{-10} \times WR^2 \times n^2 \text{ MWs}
\]  
(3.18)

Since \( \omega_o = 2\pi f_o = 120\pi \), equation (3.15) can also be written as

\[
\frac{H}{60\pi} \frac{d^2 \delta}{dt^2} = P_a \left(1 - \frac{\Delta \omega_e}{\omega_o}\right)
\]  
(3.19)

If the deviation from normal speed is negligible

\[
\frac{H}{60\pi} \frac{d^2 \delta}{dt^2} \approx P_a
\]  
(3.20)

**Mechanical Torque**

The mechanical torques of the prime movers of large generators are functions of speed.

**Unregulated machine**

For a fixed gate or valve position (i.e., when the machine is not under active governor control) the torque speed characteristic is nearly linear over a limited range at rated speed.

From the relationship between mechanical torque and power,

\[
T_m = P_m / \omega
\]  
(3.21)

From (3.21)

\[
\Delta T_m = \frac{1}{\omega_o} \Delta P_m - \frac{P_m}{\omega_o} \Delta \omega
\]  
(3.22)

For an unregulated machine \( \Delta P_m = 0 \). Therefore

\[
\Delta T_m = - \frac{P_m}{\omega_o} \Delta \omega = - \frac{P_m}{\omega_o} T_m \Delta \omega
\]

or, in per unit,

\[
\Delta T_{m\text{pu}} = - \Delta \omega_{\text{pu}}
\]  
(3.23)

This relationship is shown in Figure 3.1.

![Fig. 3.1 Turbine torque-speed characteristic of an unregulated machine.](image)
Regulated machine

In regulated machines the speed control mechanism is responsible for controlling the throttle valve to the steam turbine or the gate position in hydro turbines, and the mechanical torque is adjusted accordingly. To be stable under normal operating conditions, the torque speed characteristic of the turbine speed control system should have a "droop characteristic," i.e., a drop in turbine speed should accompany an increase in load. Such a characteristic is shown in Figure 3.2. A typical "droop" or "speed regulation" characteristic is 5% in the United States (4% in Europe).

![Fig. 3.2 Turbine torque-speed characteristic of a regulated machine.](image)

From Figure 3.2

\[ \Delta T_m = - \frac{1}{R} \Delta \omega \]  \hspace{1cm} (3.24)

Therefore, from (3.22) and (3.24),

\[ \frac{1}{\omega_o} \Delta P_m = \frac{P_m}{\omega_o^2} \Delta \omega = - \frac{1}{R} \Delta \omega \]

or

\[ \Delta P_m = \left( P_m - \frac{\omega_o^2}{R} \right) \frac{\Delta \omega}{\omega_o} \]

The above equation can be expressed in per unit as

\[ \Delta P_{m \text{ pu}} = \left( 1 - \frac{\omega_o^2}{R P_m} \right) \Delta \omega_{\text{pu}} \]

or

\[ \Delta P_{m \text{ pu}} \approx - \frac{\omega_o^2}{R P_m} \Delta \omega_{\text{pu}} = - \frac{1}{R_{\text{pu}}} \Delta \omega_{\text{pu}} \]  \hspace{1cm} (3.25)

where

\[ R_{\text{pu}} = \frac{R P_m}{\omega_o^2} \]  \hspace{1cm} (3.26)
SYNCHRONOUS MACHINE STABILITY BASICS

If the per unit regulation is to be expressed in the system base, which may be different from the machine base,

\[
R_{pu,ss} = \frac{P_s}{P_m} = \frac{\omega_o^2}{\omega_m^2} \frac{P_m}{\omega_m^2} = R_{pu} \frac{P_s}{P_m}
\]  

(3.27)

The droop characteristic shown in Figure 3.2 is obtained in the speed control system with the help of feedback. Equation (3.25) describes the steady state regulation characteristics. Transient characteristics depend on the dynamic response characteristics of the turbine control system involving various time lags in the feedback elements of the speed control system and in the steam paths.

Small Disturbance Performance of Unregulated Synchronous Machine System

In this section we investigate the general behavior of an electric power system when subjected to small disturbances. Such disturbances are always present during normal operation of power systems. The response of a power system following a disturbance is oscillatory. For stable and satisfactory operation, oscillations must damp out and the system must return to a steady state in a reasonably short period of time. Note that if the system survives a large disturbance such as a system fault followed by opening of one or more transmission lines, the system response, after the initial impact of the disturbance is over, is essentially determined by the small disturbance performance of the system.

A good qualitative picture of the basic system dynamic behavior can be obtained by employing the simplest dynamic model of the power system. In this model the power system is reduced, retaining only the major synchronous machines in the area under investigation. The number of synchronous machines retained would depend on the relative impact of these machines on the particular study. In a simplified analysis intended to reveal the essential dynamic characteristics of the system, the synchronous machines may be represented by constant voltage magnitude behind transient reactance (the so-called classical model).

A drawback of the simplified representation is that it excludes machine damping and, therefore, system simulations using this model fail to reveal the system damping behavior. Damping is inherently small in electric power systems. The small amount of damping originates mainly in the synchronous machines due to the electromagnetic interaction, and to some extent in specific load types. When damping is definitely known to exist, an approximate damping term may be included in the swing equation. Synchronous machine damping can be significantly affected, sometimes to the detriment of satisfactory system performance, due to the action of automatic voltage regulators and governors. Analyses of synchronous machine damping and the effect of excitation control will be discussed in detail in Chapter 6.

For small disturbance analysis the system equations may be linearized around an operating point and the stability and dynamic response characteristics of the resulting linear system may be investigated using the techniques discussed in Chapter 2.

Single machine connected to infinite bus

Consider a synchronous machine connected to an infinite bus through an impedance as depicted in Figure 3.3. The equation of motion of the synchronous machine is given by
\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P_m = P_e - K_d \frac{d\delta}{dt}
\] (3.28)

where \( P_m \) and \( P_e \) are the mechanical input and electrical outputs, respectively, and \( K_d \) is the damping coefficient.

\[ \frac{\Delta P_m}{\omega_o} = \frac{d\Delta \delta}{dt} = -K \Delta \delta - \frac{K_d}{\omega_o} \frac{d\Delta \delta}{dt} \] (3.29)

where \( K \) is the synchronizing power coefficient given by

\[
K = \frac{dP_e}{d\delta} = \frac{E_1 E_2}{X} \cos \delta_o
\] (3.30)

Equation (3.29) can be rearranged as

\[
\frac{d^2 \Delta \delta}{dt^2} + \frac{K_d}{2H} \frac{d\Delta \delta}{dt} + \frac{\omega_o K}{2H} \Delta \delta = 0
\] (3.31)

The characteristics roots of the system represented by equation (3.31) are

\[
\lambda_{1,2} = -\zeta \pm \sqrt{1 - \zeta^2} \omega_n
\] (3.32)

where the natural frequency \( \omega_n \) is

\[
\omega_n = \sqrt{\frac{\omega_o K}{2H}}
\] (3.33)

and the damping coefficient \( \zeta \) is

\[
\zeta = \frac{K_d}{\sqrt{8H \omega_o K}}
\] (3.34)
Since the damping is usually small ($\zeta \ll 1$), the response is oscillatory with an angular frequency of oscillation essentially the same as that given by equation (3.33). From the characteristic roots it is clear that for stability $K > 0$ and $K_d > 0$. If either of these quantities is negative, the system is unstable.

**Modes of Oscillation of an Unregulated Multi-Machine System**

Consider a power system with $n$ synchronous machines. Assume that the system has been reduced retaining only the machine internal buses.

The equation of motion of machine $i$ is

$$\frac{2H_i}{\omega_o} \frac{d^2 \delta_i}{dt^2} = P_{ai} = P_{mi} - P_{ei}$$  \hspace{1cm} (3.35)

In this analysis, the damping coefficients $K_{di}$ are assumed to be zero. Since the damping coefficients are usually small and positive, neglecting them will not alter the modes of oscillation significantly. However, the computational burden will be reduced considerably.

The expression for $P_e$ has been derived Chapter 1 and is given by

$$P_{ei} = \sum_{j=1}^{n} E_i E_j \left( G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij} \right) \ i = 1, 2, \cdots, n$$  \hspace{1cm} (3.36)

Linearizing equations (3.35) and (3.36),

$$\frac{2H_i}{\omega_o} \frac{d^2 \Delta \delta_i}{dt^2} = -\Delta P_{ei}$$  \hspace{1cm} (3.37)

and

$$\Delta P_{ei} = \sum_{j=1}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right) \Delta \delta_{ij}$$  \hspace{1cm} (3.38)

Since $\Delta \delta_{ij} = \Delta \delta_i - \Delta \delta_j$, (3.38) can be written as

$$\Delta P_{ei} = \sum_{j=1}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right) \Delta \delta_i$$

$$- \sum_{j=1}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right) \Delta \delta_j$$

$$= \sum_{j=1}^{n} K_{ij} \Delta \delta_j$$  \hspace{1cm} (3.39)

where $K_{ij} = \frac{\partial P_i}{\partial \delta_j}$, and can be called the synchronizing power coefficient between machines $i$ and $j$.

Note that in this particular representation
\[
\sum_{j=1}^{n} K_{ij} = 0 \quad (3.40)
\]

From (3.37) and (3.39)
\[
\frac{2H_i}{\omega_o} \frac{d^2 \Delta \delta_i}{dt^2} = -\sum_{j=1}^{n} K_{ij} \Delta \delta_j \quad i = 1, 2, \ldots, n \quad (3.41)
\]

Equation (3.41) can be written in matrix form as
\[
\mathbf{H} \frac{d^2}{dt^2} \Delta \mathbf{\delta} = \mathbf{P_s} \Delta \mathbf{\delta} \quad (3.42)
\]
where
\[
\mathbf{H} = \text{diag} \left( \frac{2H_1}{\omega_o}, \frac{2H_2}{\omega_o}, \ldots, \frac{2H_n}{\omega_o} \right)
\]

\[
\mathbf{P_s} = \begin{bmatrix}
-K_{11} & -K_{12} & \cdots & -K_{1n} \\
-K_{21} & -K_{22} & \cdots & -K_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-K_{n1} & -K_{n2} & \cdots & -K_{nn}
\end{bmatrix}, \quad \text{and} \quad \Delta \mathbf{\delta} = \begin{bmatrix}
\Delta \delta_1 \\
\Delta \delta_2 \\
\vdots \\
\Delta \delta_n
\end{bmatrix}
\]

From (3.42)
\[
\frac{d^2}{dt^2} \Delta \mathbf{\delta} = \mathbf{H}^{-1} \mathbf{P_s} \Delta \mathbf{\delta} = \mathbf{K} \Delta \mathbf{\delta} \quad (3.43)
\]

Assume a solution \( \Delta \mathbf{\delta} = e^{\lambda t} \mathbf{e} \)

Then
\[
\lambda^2 e^{\lambda t} = \mathbf{K} e^{\lambda t}
\]
from which
\[
\left[ \lambda^2 \mathbf{I} - \mathbf{K} \right] e^{\lambda t} = 0 \quad (3.44)
\]
Equation (3.44) has a nontrivial solution only if
\[
\lambda^2 \mathbf{I} - \mathbf{K} = 0 \quad (3.45)
\]

Therefore, for stability all the roots, \( \lambda^2 \), of equation (3.45), i.e., the characteristic values (or the eigenvalues) of the matrix \( \mathbf{K} \), must be real and negative. If any of the roots is real and positive, the system is unstable.

Actually, since the matrix \( \mathbf{K} \) is singular (the columns add up to zero, since \( \sum K_{ij} = 0 \) ) one of the eigenvalues is always zero, since there are \((n-1)\) independent \( \Delta \delta 's \). This can be avoided by taking one machine as a reference. If machine \( n \) is taken as a reference, the modified \( \mathbf{K} \) matrix would be obtained by deleting the \( n \)th column and subtracting the \( n \)th row from every other row. The modified \( \mathbf{K} \) matrix would then become
For stability, the \((n-1)\) eigenvalues of \(K_1\) must be real and negative. This means that the system represented by equation (3.43) has \(2(n-1)\) imaginary roots or \((n-1)\) conjugate pairs. Therefore, the system has \((n-1)\) natural frequencies of oscillations.

**Problem**

Derive the condition for stability of the two machine system shown in Figure 3.4.

The following steps are involved in computing the modes of oscillation of a multi-machine system. The system shown in Figure 3.5 is used for the purpose of illustration.

1. At a given load level, the bus voltages (magnitudes and angles) are first obtained from a power flow solution. Since \(P, Q, V\) and \(\theta\) are known at the machine terminals, the internal voltages and angles of the machines can be computed as shown in Chapter 1.
2. Since non-synchronous loads do not significantly influence the modes of oscillation, very little error will be incurred by representing loads as constant impedances (or admittances). Therefore, all loads are converted into equivalent admittances. This can be done as follows:

\[ P_L - jQ_L = V^* I \]  \hspace{1cm} (3.46)

where \( V \) is the bus voltage and \( I \) is the current into the load given by

\[ I = V Y_L \] \hspace{1cm} (3.47)

where \( Y_L \) is the equivalent load admittance.

From (3.46) and (3.47)

\[ Y_L = \frac{I}{V} = \frac{P - jQ}{V^* V} = \frac{P - jQ}{|V|^2} \]

from which

\[ Y_L = \frac{P}{|V|^2} - j \frac{Q}{|V|^2} \] \hspace{1cm} (3.48)

3. The network is then reduced, retaining only the machine internal buses. This can be done in several ways. A general method suitable for stability computations discussed in later chapters will be illustrated here.

The network equation in terms of the bus admittance matrix is formed by treating the machine internal buses as additional system buses as shown in equation (3.49).

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_N
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{jx'_{d1}} & -1 & & & \\
\frac{1}{jx'_{d2}} & \frac{1}{jx'_{d1}} & -1 & & \\
& \frac{1}{jx'_{d3}} & \frac{1}{jx'_{d2}} & \ddots & \\
& & \ddots & \ddots & -1 \\
& & & \frac{1}{jx'_{d2}} & \frac{1}{jx'_{d1}}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
\vdots \\
E_N
\end{bmatrix}
\]

\[ Y_1, Y_2, \ldots \] are the elements of the network admittance matrix. Since the loads have been converted into equivalent admittances and the machine terminal buses are now passive buses due to the inclusion of the machine internal buses into the system network, the injected currents at all the buses except at those corresponding to the machine internal buses are zero.
For a system of $m$ generators and $n$ buses (3.49) can be written as

$$
\begin{bmatrix}
I
\end{bmatrix}
= \begin{bmatrix}
Z_M^{-1} & -Z_M^{-1} \\
-Z_M^{-1} & Y_1 + Z_M^{-1} Y_2 \\
\end{bmatrix}
\begin{bmatrix}
E \\
V
\end{bmatrix}
$$

(3.50)

where

$Z_M = m \times m$ diagonal matrix composed of the machine internal impedances

$I =$ vector of generator currents

$E, V =$ vectors of machine internal voltages and network bus voltages, respectively

$Y_1, Y_2, \text{etc.} =$ submatrices obtained by partitioning the network admittance matrix $Y_N$

From (3.50) we can compute the network bus voltages, which may be needed in some stability studies, as

$$
V = Y_{MN}^{-1} \begin{bmatrix}
Z_M^{-1} \\
0
\end{bmatrix} E
$$

(3.51)

where

$$
Y_{MN} = \begin{bmatrix}
Y_1 + Z_M^{-1} Y_2 \\
Y_3 \\
Y_4
\end{bmatrix}
$$

From (3.50) and (3.51)

$$
I = Z_M^{-1} E + \begin{bmatrix}
-Z_M^{-1} \\
0
\end{bmatrix} Y_{MN}^{-1} \begin{bmatrix}
Z_M^{-1} \\
0
\end{bmatrix} E
$$

(3.52)

After carrying out the operations indicated in (3.52), we obtain

$$
I = Z_M^{-1} E - Z_M^{-1} \left[ Y_{RN} + Z_M^{-1} \right]^{-1} Z_M^{-1} E
$$

(3.53)

where $Y_{RN}$ is the reduced network admittance matrix (see Chapter 1).

$$
Y_{RN} = Y_1 - Y_2 Y_4^{-1} Y_3
$$

Equation (3.53) can be written as

$$
I = YE
$$

(3.54)

where

$$
Y = Z_M^{-1} - Z_M^{-1} \left[ Y_{RN} + Z_M^{-1} \right]^{-1} Z_M^{-1}
$$

$Y_{MN}$ is, in general, very sparse and in a large scale stability computation it may be more convenient and desirable to obtain $I$ directly from (3.52) by exploiting this sparsity.

From (3.52) (as well as (3.53)) it is evident that the same results as obtained from the above procedure can also be obtained as follows:
Convert each of the machine internal voltages in series with the transient reactance into a current source, \( I = E / jx'_d \), in parallel with \( jx'_d \); include the shunts thus obtained into the network admittance matrix and obtain the machine terminal voltages from the relation \( V = Y_{MN}^{-1} I^1 \), where \( I^1 \) is the vector whose elements are \( E_i / jx'_d \); then obtain the machine currents from the relation 
\[
I_i = (E_i - V_i) / jx'_d \quad i = 1, 2, \ldots, m .
\]
Alternatively, for each generator 
\[ I = \frac{E - V}{jx'_d} \]
Therefore, for the entire network we can write 
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_M^{-1} \\ 0 \end{bmatrix} E - \begin{bmatrix} Z_M^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} V \]
(3.55)
The network equation 
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Y_N V \quad (3.56)
\]
From (3.55) and (3.56) 
\[
\begin{bmatrix} Z_M^{-1} \\ 0 \end{bmatrix} E = Y_{MN} V
\]
from which (3.51) follows.
Substituting (3.51) in (3.55) and simplifying, we obtain (3.52), from which (3.54) follows.
4. The elements \( K_{ij} \) are computed from equation (3.39) and the matrix \( K \) is formed.
5. The eigenvalues of the matrix \( K \) are then computed using an eigenvalue computing subroutine. For stability, all the eigenvalues must be real and negative.
6. The square roots of the negative of these eigenvalues are the natural frequencies of oscillations.

An alternative solution for the modes of oscillation of an unregulated multi-machine system

Equation (3.43) can be written, using variables \( x_1, x_2, \cdots \) to denote \( \Delta \delta_1, \Delta \delta_2, \cdots \), as 
\[
\frac{d^2}{dt^2} x = Kx
\]
(3.57)
Using a transformation 
\[ x = Mz \]
(3.58)
where \( M \) is the modal matrix (see Chapter 2),
we have 
\[
\frac{d^2}{dt^2} z = M^{-1} K Mz = Dz
\]
(3.59)
where 
\[ D = \text{diag} \left( \lambda_1, \lambda_2, \cdots \right) \]
\( \lambda_1, \lambda_2, \text{ etc.} \) are the eigenvalues of the matrix \( \mathbf{K} \).

Therefore

\[
\frac{d^2 z_1}{dt^2} = \lambda_1 z_1, \quad \frac{d^2 z_2}{dt^2} = \lambda_2 z_2, \quad \text{etc.} \tag{3.60}
\]

For stability \( \lambda_1, \lambda_2, \text{ etc.} \) must be real and negative.

The solutions to the above second order equations are of the form

\[
z_1 = a_1 \cos(\omega_1 t - \phi_1), \quad z_2 = a_2 \cos(\omega_2 t - \phi_2), \quad \text{etc.}
\]

where

\[
\omega_1 = \sqrt{-\lambda_1}, \quad \omega_2 = \sqrt{-\lambda_2}, \quad \text{etc.}
\]

The constants \( a' \)'s and \( \phi' \)'s depend on the initial condition. For example, if the initial perturbation is such that \( \dot{x}_{10} = \dot{x}_{20} = \cdots = 0 \), then

\[
\frac{d}{dt} z_0 = \mathbf{M}^{-1} \frac{d}{dt} x_0 = 0
\]

which yields

\[
\phi_1 = \phi_2 = \cdots = 0
\]

Therefore

\[
z = \mathbf{R}(t) \mathbf{a}
\]

where

\[
\mathbf{R}(t) = \text{diag}(\cos \omega_1 t, \cos \omega_2 t, \cdots)
\]

\[
\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots \end{bmatrix}
\]

From the above

\[
\mathbf{a} = \mathbf{z}_0 = \mathbf{M}^{-1} \mathbf{x}_0
\]

which yields

\[
\mathbf{x} = \mathbf{M} \mathbf{R}(t) \mathbf{M}^{-1} \mathbf{x}_0 \tag{3.61}
\]

The general solution can be written as

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & \cdots \\ e_{21} & e_{22} & e_{23} & \cdots \\ e_{31} & e_{32} & e_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_1 \cos(\omega_1 t - \phi_1) \\ a_2 \cos(\omega_2 t - \phi_2) \\ a_3 \cos(\omega_3 t - \phi_3) \\ \vdots \end{bmatrix} \tag{3.62}
\]

The relationship between the various modes and mode shapes and the system response can be seen in equation (3.62). For example, if \( e_{11} = 0 \), the first mode is absent in the variable \( x_1 \). Also, if \( e_{21} \) and \( e_{31} \) are of opposite sign, the first mode is in phase opposition in variables \( x_2 \) and \( x_3 \). Thus, in addition to providing information on system stability, a knowledge of the modes and the mode shapes provide considerable insight into the character of the system response.
Division of Suddenly Applied Load (or Generation Loss) Among Generators in the System

A suddenly applied load (or a sudden generation loss) creates an unbalance between generation and load and a transient results. The system passes through an oscillatory state before settling to a new steady-state condition. Following the impact, the amount of load picked up by each generator at various instants of time can be estimated from a knowledge of the system and the various control parameters. Although the following analysis is approximate, it throws much light on the behavior of the system immediately following a load (or generation) impact. For the purpose of our analysis we assume that the generators can be represented by the classical model (i.e., by constant voltage magnitude behind transient reactance) and that all the non-generator buses except the one where the load impact occurs have been eliminated.

Referring to the network shown in Figure 3.6, where a load $P_L$ is suddenly applied at bus $n$, the equation for power into each node is obtained from

$$P_{ei} = \sum_{j=1}^{n} E_i E_j \left( G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij} \right) \quad i = 1, 2, \ldots, n$$

(3.63)

It is assumed that the applied load $P_L$ has a negligible reactive component.

![Fig. 3.6 Network showing load impact at bus n.](image)

Since the impact load $P_L$ is small compared with the total load, and we are concerned with the change in the loading of the system generators (load pick-up), we can linearize the system equations. Linearizing (3.63),

$$\Delta P_{ei} = \sum_{j=1}^{n} \frac{\partial P_{ei}}{\partial \delta_j} \Delta \delta_j \quad i = 1, 2, \ldots, n$$

(3.64)

where the $\frac{\partial P_{ei}}{\partial \delta_j}$'s are the synchronizing power coefficients given by, neglecting line conductances,

$$\frac{\partial P_{ei}}{\partial \delta_i} = \sum_{j=i}^{n} E_i E_j B_{ij} \cos \delta_{ij}$$

(3.65)

and

$$\frac{\partial P_{ei}}{\partial \delta_j} = -E_i E_j B_{ij} \cos \delta_{ij}$$

(3.66)

Also

$$P_L = -\Delta P_{en}$$

(3.67)
From equations (3.64) through (3.67), it follows that

\[ \sum_{i=1}^{n} \Delta P_{ei} = 0 \]  
(3.68)

or

\[ \sum_{i=1}^{n-1} \Delta P_{ei} = -\Delta P_{en} = P_L \]  
(3.69)

**Load pick-up at \( t = 0^+ \)**

At the instant immediately following the load impact, i.e., at \( t = 0^+ \), \( \Delta \delta_i = 0 \) for all generators, since the rotor angles cannot change instantaneously because of the rotor inertias. The only change in the angle at \( t = 0^+ \) is that of the load bus voltage. Therefore, from (3.64),

\[ \Delta P_{ei} = \frac{\partial P_{ei}}{\partial \delta_n} \Delta \delta_n = -E_i E_n B_m \cos \delta_m \Delta \delta_n \quad i = 1, 2, \ldots, n-1 \]  
(3.70)

From equations (3.69) and (3.70), we have

\[ \sum_{i=1}^{n-1} \Delta P_{ei} = P_L = \sum_{i=1}^{n-1} \frac{\partial P_{ei}}{\partial \delta_n} \Delta \delta_n \]  
(3.71)

and

\[ \Delta P_{ei} = \frac{\partial P_{ei}}{\partial \delta_n} \frac{P_L}{\sum_{i=1}^{n-1} \frac{\partial P_{ei}}{\partial \delta_n}} \quad i = 1, 2, \ldots, n-1 \]  
(3.72)

Therefore, the load impact \( P_L \) at bus \( n \) at \( t = 0^+ \) is shared by the synchronous generators according to their synchronizing power coefficients with respect to the bus \( n \). Thus, the machines electrically close to the point of impact will pick up the greater share of the load regardless of their size. At \( t = 0^+ \), the source of energy supplied by the generators is the energy stored in their magnetic field.

Due to the sudden increase in the output power the generators will start decelerating. From the swing equations, it is apparent that the initial deceleration of a particular machine \( i \) will be dependent on the synchronizing power coefficient \( \frac{\partial P_{ei}}{\partial \delta_n} \) and the inertia constant \( H_i \). Thereafter, the machines follow an oscillatory motion governed by the swing equations.

**Average behavior prior to governor action**

As the machines decelerate the individual machine speeds will drop and governor action will be initiated following a time lag. We now estimate the system behavior during the period from \( t = 0^+ \) to the time when the governor action becomes appreciable. During this period the system as a whole will be retarding. Although the individual machines will initially be retarding at different rates, if the system remains stable, they will acquire the same mean retardation after the initial transients have decayed.
The incremental swing equations governing the motion of the machines are given by

\[
\frac{2H_i}{\omega_o} \frac{d\Delta \omega_i}{dt} = -\Delta P_{ei} \quad i = 1, 2, \ldots, n - 1
\]  

(3.73)

Assuming that the individual machines have acquired the same mean retardation \(d\Delta \omega / dt\), equation (3.73) for the individual machines can be added together to give

\[
\frac{2\sum_{i=1}^{n-1} H_i}{\omega_o} \frac{d\Delta \omega}{dt} = -\sum_{i=1}^{n-1} \Delta P_{ei} = -P_L
\]  

(3.74)

Substituting (3.74) into (3.73) we have

\[
\Delta P_{ei} = \frac{P_L}{\sum_{i=1}^{n-1} H_i} H_i
\]  

(3.75)

Equation (3.75) shows that the (mean) increase in output of the individual generators in response to an impact load is proportional to their inertia constants.

Equation (3.73) can also be written as

\[
\frac{2}{\omega_o} \frac{d}{dt} (\Delta \omega_i H_i) = -\Delta P_{ei} \quad i = 1, 2, \ldots, n - 1
\]  

(3.76)

Adding the equations corresponding to the individual machines, we have

\[
\frac{2}{\omega_o} \frac{d}{dt} \left( \sum_{i=1}^{n-1} \Delta \omega_i H_i \right) = -\sum_{i=1}^{n-1} \Delta P_{ei} = -P_L
\]  

(3.77)

Equation (3.77) can be written as

\[
\frac{2\sum_{i=1}^{n-1} H_i}{\omega_o} \frac{d}{dt} \left( \frac{\sum_{i=1}^{n-1} \Delta \omega_i H_i}{\sum_{i=1}^{n-1} H_i} \right) = -P_L
\]  

(3.78)

Comparing equation (3.78) with equation (3.74), we can define the mean angular speed deviation as the weighted mean

\[
\Delta \bar{\omega} = \frac{\sum_{i=1}^{n-1} \Delta \omega_i H_i}{\sum_{i=1}^{n-1} H_i}
\]  

(3.79)

Similarly, we can define a mean angle deviation
In the derivation of equation (3.75), which shows the change in electrical output of each machine to be proportional to their respective inertias, the assumption was made that the individual machines in the system reached the same mean retardation a short time after the load impact occurred and before appreciable governor action took place. In a large system the time needed to reach a uniform mean retardation may be considerable. Generators electrically remote from the point of load impact may not "feel" the effect of the impact and, therefore, experience any speed change in order to participate in the inertial load pick-up, until after an appreciable length of time. By then, the governor action would start modifying the mechanical input of the local generators in response to their speed change that has been sustained sufficiently long to overcome the time lag. It follows, therefore, that in a large interconnected system, the generation pick-up following a sudden load addition (or generator loss) will never be strictly in accordance with the inertial distribution of the entire system. In estimating (inertial) pick-up during the brief period immediately following a load impact, inertias of generators electrically remote from the point of load impact should be excluded, since these generators will not participate significantly in the inertial pick-up during this period. Governor action will start modifying system response well before the inertias of the remote machines can exert any appreciable effect in the generation pick-up.

In order for the governor of a generator to respond, the speed deviation has to exceed the governor dead band. Since the generators electrically close to the bus at which the load impact occurred would experience the largest speed deviation initially, these will respond first, by overcoming the governor dead band and valve set point nonlinearities. The generators further away would experience a more gradual drop in speed, and in the event the speed change does not exceed the governor dead band, governor action will not be initiated. The actual pick-up by individual generators following the initiation of governor action will depend on the total system capacity, the amount of load addition (or lost generation), the relative locations of the generators and their governor-turbine characteristics. The change in mechanical input of a particular generator due to governor action is obtained from

$$\Delta P_{mi} = \frac{\Delta \omega P_i}{\omega_o R}$$

where

- $P_i =$ MW capacity of generator $i$
- $R =$ system regulation in pu (= 0.05)
- $\omega_o =$ nominal system frequency in rad/sec

If $\Delta f$ (Hz) is the governor dead band, i.e., the governor is insensitive to frequency deviation of less than $\Delta f$ Hz, the condition for all the generators in the interconnected system to respond due to governor action is

$$P_L \geq \sum \frac{\Delta f P_i}{f_o R}$$

where $P_L$ is the amount of load addition (or lost generation).
Therefore, if the total generation of the interconnected system ($\Sigma P_i$) is greater than ($f_o/\Delta f$) $R P_L$, then some of the generators electrically remote from the bus of load impact will not experience the required frequency change of $\Delta f$ Hz in order for their governors to respond. Of course, some of the generators in the system may not be under active governor control. This will have to be accounted for in the above analysis.

It should be noted that the frequency change at some of the generator buses will always exceed the governor dead band no matter how small the load impact is, since the mismatch between generation and load created by the load impact, if left uncorrected, will continually increase the frequency deviation from normal.

The transition from the inertial generation pick-up to the generation pick-up dictated by governor action is oscillatory, the frequency of oscillation being a function of the machine inertia, governor speed regulation and servomotor time constant. Eventually, the speed deviation and the changes in the tie-line flows will be detected by the AGC system and the system generation would be adjusted according to a set criterion.

**Transient Stability by Equal Area Criterion**

In power system stability studies the term transient stability usually refers to the ability of the synchronous machines to remain in synchronism during the brief period following a large disturbance. In a large disturbance, system nonlinearities play a dominant role. In order to determine transient stability or instability following a large disturbance, or a series of disturbances, it is usually necessary to solve the set of nonlinear equations describing the system dynamics. Conclusions about stability or instability can then be drawn from an inspection of the solution. Since a formal solution of the equations is not generally possible, an approximate solution is usually obtained by a numerical technique.

For one and two machine systems, assuming that the machines can be represented by the classical model, i.e., by constant voltage behind transient reactance, a simple graphical method known as the equal area criterion can be utilized to assess transient stability. When a large interconnected system is subjected to a large disturbance, as a rule, it splits into a small number of groups of machines which swing against one another while the machines within each group swing together. Frequently there are only two major groups and the general behavior of the system is similar to that of a two machine system. In such situations, for the purpose of approximate analyses, the large system may be replaced by an equivalent two machine system.

Consider a single synchronous machine connected to an infinite bus as shown in Figure 3.7.

\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P'' = P_m - P_e
\]  

which can be written as
\[
\frac{d^2 \delta}{dt^2} = \frac{\omega_o}{2H} P_a \tag{3.84}
\]

Multiplying both sides of (3.84) by \(2 \frac{d\delta}{dt}\), we have

\[
2 \frac{d\delta}{dt} \frac{d^2 \delta}{dt^2} = \frac{\omega_o}{2H} P_a 2 \frac{d\delta}{dt} \]

or

\[
\frac{d}{dt} \left( \frac{d\delta}{dt} \right)^2 = \frac{\omega_o}{H} P_a \frac{d\delta}{dt} \]

or

\[
\left( \frac{d\delta}{dt} \right)^2 = \frac{\omega_o}{H} \int_{\delta_o}^\delta P_a \, d\delta \tag{3.85}
\]

Integrating (3.85), we obtain

\[
\left( \frac{d\delta}{dt} \right)^2 = \frac{\omega_o}{H} \int_{\delta_o}^\delta P_a \, d\delta \tag{3.86}
\]

In equation (3.86), \(d\delta/dt\) is the relative speed of the machine with respect to a synchronously rotating reference frame. The initial value of \(d\delta/dt\) is zero. Following a disturbance \(P_a\) has a positive value and the machine accelerates. For stability, \(P_a\) must reverse sign and \(d\delta/dt\) must come to zero before \(P_a\) reverses sign again. Therefore, for stability, there is a \(\delta\) such that \(P_a(\delta) \leq 0\) and

\[
\int_{\delta_o}^\delta P_a \, d\delta = 0 \tag{3.87}
\]

In the limit

\[
P_e (\delta_{\text{max}}) = 0 \text{ and } \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} P_a \, d\delta = 0 \tag{3.88}
\]

If the accelerating power is plotted as a function of \(\delta\), equation (3.88) can be interpreted as the area under the curve between \(\delta_o\) and \(\delta_{\text{max}}\). The net area under the curve adds to zero (the positive and negative parts of the area are equal; hence the name -- equal area criterion).

Consider the case when \(P_m\) is constant. For the system shown in Figure 3.7, \(P_e\) is given by

\[
P_e = \frac{E_1 E_2}{X} \sin \delta \tag{3.89}
\]

The value of \(X\) would depend on the network condition, i.e. whether pre-fault, fault or post-fault. The power angle curves corresponding to pre-fault, fault and post-fault conditions are shown in Figure 3.8. By applying the criterion of equation (3.88), it can be seen that the system is stable for a fault clearing angle of up to \(\delta_c\), when the areas \(A_1\) and \(A_2\) are equal. For a clearing angle greater than \(\delta_c\) the system is unstable. The angle \(\delta_c\) is called the critical clearing angle and the corresponding maximum rotor angle reached is \(\delta_{\text{m}}\).
Problem

Show that the critical clearing angle $\delta_c$ shown in Figure 3.8 is given by

$$\cos \delta_c = \frac{1}{r_2 - r_1} \left[ \frac{P_m}{P_M} (\delta_m - \delta_o) + r_2 \cos \delta_m - r_1 \cos \delta_o \right]$$

where

- $P_M = \text{peak of the pre-fault power-angle curve}$
- $r_1 = \text{ratio of the peak of the power-angle curve of the faulted network to } P_M$
- $r_2 = \text{ratio of the peak of the power-angle curve of the post-fault network to } P_M$

The equal area criterion provides information on the critical clearing angle. However, the corresponding clearing time is of primary importance. The clearing time must be obtained from a time solution of the swing equation. A graphical solution of the swing equation is illustrated below.

From equation (3.86), the relative angular speed of the machine is given by

$$\frac{d\delta}{dt} = \omega = \sqrt{\frac{\omega_o}{H} \int_{\delta_o}^{\delta} P_a \ d\delta}$$

(3.90)

Graphically, the integral in equation (3.90) is the area under the curve of $P_a$ against $\delta$, or between the curves of $P_m$ and $P_e$ against $\delta$. By rearranging equation (3.90) and integrating, we obtain

$$t = \int_{\delta_o}^{\delta} \frac{d\delta}{\omega} = \int_{\delta_o}^{\delta} \frac{d\delta}{\sqrt{\frac{\omega_o}{H} \int_{\delta_o}^{\delta} P_a \ d\delta}}$$

(3.91)

Evaluation of this integral gives $t$ as a function of $\delta$, which is the swing curve.
In general, a formal solution of equation (3.91) is not possible. If $P_a$ is constant, a formal solution can be obtained. Then

$$t = \sqrt{\frac{4H(\delta - \delta_o)}{\omega_o P_a}}$$

(3.92)

The integral of equation (3.91) can be evaluated graphically. This may be done by plotting a curve of $1/\omega$ against $\delta$ and finding the area under the curve as a function of $\delta$. Some difficulty will be encountered in determining the area under the curve for values of $\delta$ near the initial value $\delta_0$ and the maximum value $\delta_m$, because at these values of $\delta$, $\omega$ is zero. This difficulty may be avoided by assuming $P_a$ to be constant over a small range of $\delta$ (until the curve of $1/\omega$ comes back on scale) and by using equation (3.92) to compute the time for the machine to swing through this range of $\delta$.

**Effect of damping**

In the system model used in the previous section (equation (3.83)), damping was neglected. In a stable system when the machine rotor reaches the maximum angle, $d\delta/dt = 0$, and the net torque on the rotor is retarding. The rotor will therefore start to swing backward and go past the post-fault equilibrium point and continue until it reaches the point where $d\delta/dt = 0$, and the total area between the post-fault power-angle curve and the input line is zero, i.e., the area above the input line is equal to the area below the input line (the same equal area criterion applies). At this point the net torque on the rotor is accelerating and therefore the rotor will start to swing forward. In the absence of damping the rotor will continue to swing forward and backward at constant amplitude.

The presence of positive damping (see equation (3.28)) will provide a component of electrical power (or torque) in proportion to speed, i.e., a component of power that is positive when the angle is increasing and vice versa. This will modify the power-angle curve in accordance with the damping term in (3.28). It can be seen that, including the effect of damping, the area between the power-angle curve and the input line becomes smaller in each successive swing, thereby allowing the rotor to eventually settle at the post-fault equilibrium point. Note that if the damping is assumed to be negative, i.e., the component of electrical power decreasing with increase in angle, the area between the power-angle curve and the input line will increase in each subsequent swing, thereby causing instability.

**Equal area criterion for a two-machine system**

A two-machine system can be reduced to an equivalent one machine-infinite bus system. The equivalent system can be derived as follows.

The swing equations of the two machines are

$$\frac{2H_1}{\omega_o} \frac{d^2\delta_1}{dt^2} = P_{a1} = P_{m1} - P_{e1}$$  

(3.93)

$$\frac{2H_2}{\omega_o} \frac{d^2\delta_2}{dt^2} = P_{a2} = P_{m2} - P_{e2}$$  

(3.94)

Equations (3.93) and (3.94) can be combined as
\[
\frac{d^2\delta_{12}}{dt^2} = \frac{\omega_o P_{a1}}{2H_1} - \frac{\omega_o P_{a2}}{2H_2}
\]  

(3.95)

where

\[
\delta_{12} = \delta_1 - \delta_2
\]

Equation (3.95) can be written as

\[
\frac{2}{\omega_o} \frac{H_1 H_2}{H_1 + H_2} \frac{d^2\delta_{12}}{dt^2} = \frac{H_1 P_{a1} - H_2 P_{a2}}{H_1 + H_2}
\]

(3.96)

or

\[
\frac{2H}{\omega_o} \frac{d^2\delta}{dt} = P_a = P_m - P_e
\]

(3.97)

where

\[
H = \frac{H_1 H_2}{H_1 + H_2}
\]

(3.98)

is the equivalent inertia constant, and

\[
\delta = \delta_{12}
\]

The equivalent mechanical input,

\[
P_m = \frac{H_2 P_{m1} - H_1 P_{m2}}{H_1 + H_2}
\]

(3.99)

and the equivalent electrical output,

\[
P_e = \frac{H_2 P_{e1} - H_1 P_{e2}}{H_1 + H_2}
\]

(3.100)

As in the one-machine case, the expression for equal area criterion for the two-machine case is given by

\[
\left( \frac{d\delta_{12}}{dt} \right)^2 = \frac{\omega_o}{H_1 H_2} \int_{\theta_1}^{\theta_2} (H_2 P_{a1} - H_1 P_{a2}) d\delta_{12} = 0
\]

(3.101)

If network resistance is neglected

\[
P_{a2} = -P_{a1},
\]

and equation (3.101) reduces to

\[
\left( \frac{d\delta_{12}}{dt} \right)^2 = \frac{\omega_o}{H_1 H_2} \int_{\theta_1}^{\theta_2} P_{a1} d\delta_{12} = \frac{\omega_o}{H} \int_{\theta_1}^{\theta_2} P_{a1} d\delta_{12} = 0
\]

(3.102)

**Transient Stability by Liapunov’s Method**

As has been pointed out in Chapter 2, the application of Liapunov’s method in multi-machine transient stability studies involves mathematical compromises and, therefore, the method is not suitable for practical use. However, in a simple one or two machine system, when the machines
SYNCHRONOUS MACHINE STABILITY BASICS

are represented by the classical model, the method can be used conveniently without sacrificing mathematical rigor. Actually, using the transient energy as the Liapunov function, one would arrive at a stability criterion which is the same as the equal area criterion.

With $\delta = x_1$ and $d\delta/dt = x_2$, equation (3.28) can be written as

$$\frac{dx_1}{dt} = x_2$$

$$\frac{2H}{\omega_o} \frac{dx_2}{dt} = P_m - \frac{E_1E_2}{X} \sin x_1 - \frac{K_d}{\omega_o} x_2$$

Equilibrium points are obtained by putting the right hand side of (3.103) equal to zero. Thus, one equilibrium point is at $x_1 = \theta$, $x_2 = 0$ and a second one at $x_1 = \pi - \theta$, $x_2 = 0$, where

$$\theta = \sin^{-1} \left( \frac{P_m X}{E_1E_2} \right)$$

As will be seen later, the first equilibrium point is stable, and the second one is unstable.

Shifting the origin to the stable equilibrium point by a change of variable, (3.103) can be written as

$$\frac{dx_1}{dt} = x_2$$

$$\frac{2H}{\omega_o} \frac{dx_2}{dt} = P_m - \frac{E_1E_2}{X} \sin(\theta + x_1) - \frac{K_d}{\omega_o} x_2$$

The equilibrium points after shifting the origin are given by (i) $x_1 = 0$, $x_2 = 0$ and (ii) $x_1 = \pi - 2\theta$, $x_2 = 0$.

We choose a Liapunov function

$$V(x_1, x_2) = \frac{1}{2} \frac{2H}{\omega_o} x_2^2 + \int_0^{x_1} \left( -P_m + \frac{E_1E_2}{X} \sin(\theta + x_1') \right) dx_1'$$

Stationary points are given by $\frac{\partial V}{\partial x_1} = 0$, $\frac{\partial V}{\partial x_2} = 0$

or

$$-P_m + \frac{E_1E_2}{X} \sin(\theta + x_1) = 0, x_2 = 0$$

These are the same as the equilibrium states of the system given by (3.104).

The matrix $H$ of the second derivative of $V$ is given by

$$H = \begin{bmatrix} \frac{E_1E_2}{X} \cos(\theta + x_1) \\ \frac{2H}{\omega_o} \end{bmatrix}$$
Provided \( \cos \theta \) is positive, \( H \) is positive definite at the equilibrium state (i) and indefinite at the equilibrium state (ii). Thus, the stationary point \( x_1 = 0, x_2 = 0 \) is a point of local minimum and the other is a saddle point.

At \( x_1 = x_2 = 0 \), \( V(x_1, x_2) = 0 \), and thus \( V \) is positive definite in the neighborhood of the origin. The total derivative of \( V \) is

\[
\dot{V} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2
\]

Substituting the expressions for \( \dot{x}_1 \) and \( \dot{x}_2 \) from (3.104)

\[
\dot{V} = -\frac{K_d}{\omega_o} x_2^2
\]

(3.106)

By Liapunov’s theorem (see Chapter 2), the equilibrium state \( x_1 = 0, x_2 = 0 \) is asymptotically stable, whereas the equilibrium state \( x_1 = \pi - 2\theta, x_2 = 0 \) is unstable. The curves given by \( V(x_1, x_2) = C \) (constant) will obviously be closed for \( C \) sufficiently small. In order to establish the region of stability around the stable equilibrium point one has to find the largest \( V = C \) closed curve.

Calling the part of the \( V \) function given by (3.105) which involves the integral, \( F(x_1) \), we find that \( F(x_1) \) has a minimum at \( x_1 = 0 \) and a maximum at \( x_1 = \pi - 2\theta \), corresponding to the stable and unstable equilibrium points, respectively. The minimum value is zero and the maximum value is

\[
- P_m (\pi - 2\theta) + \frac{2E_i E_x}{X} \cos \theta
\]

Referring to the maximum value of \( F(x_1) \) as \( l \), the curves \( V = C \) will be closed for \( C < l \) and open for \( C > l \). This can be seen by expressing \( x_2 \) in terms of \( x_1 \). Thus

\[
x_2 = \pm \sqrt{\frac{\omega_o}{H} (V - F(x_1))}
\]

(3.107)

For \( V < l \) two distinct values of \( x_2 \) will be obtained as \( x_1 \) increases until \( F(x_1) \) becomes equal to \( V \). Beyond this the values of \( x_2 \) will no longer be real. For \( V > l \) two distinct values of \( x_2 \) will be obtained for all positive \( x_1 \)'s.

If the system is operating at a stable equilibrium point and a fault occurs, \( x_1 \) and \( x_2 \) will start to increase and so will \( V(x_1, x_2) \). If the fault is cleared before \( V(x_1, x_2) \) reaches the value \( l \), the system will be stable. In other words, if the substitution of the values of \( x_1 \) and \( x_2 \) at the time of fault clearing in the expression for \( V \) results in a value of \( V \) less than \( l \), the system will be stable. An unstable condition will be indicated if \( V > l \) \( (x_1 \geq \pi - 2\theta) \) -- \( l \) determines the extent of asymptotic stability.

**Problem**

Show that the choice of the Liapunov function (3.105) and the extent of stability determined by the condition \( V \leq l \) is equivalent to the equal area criterion as discussed earlier.

**Hint:** With the Liapunov function (3.105)
SYNCHRONOUS MACHINE STABILITY BASICS

\[ l = \int_0^{\pi-2\theta} \{-P_m + \frac{E_1 E_2}{X} \sin(\theta + x'_1)\} dx'_1 \]

At fault clearance

\[ V = \frac{H}{\omega_o} x_2^2 + \int_0^{\pi-2\theta} \{-P_m + \frac{E_1 E_2}{X} \sin(\theta + x'_1)\} dx'_1 \]  
(see Fig. 3.5)

From (3.104), neglecting damping,

\[ \frac{2H}{\omega_o} \frac{dx_2}{dx_1} = \frac{P_m - \frac{E_1 E_2}{X_F} \sin(\theta + x_1)}{x_2} \], where \( X_F \) = line reactance during fault.

Integrating

\[ \frac{H}{\omega_o} x_2^2 = \int_0^{\pi-2\theta} \{P_m - \frac{E_1 E_2}{X_F} \sin(\theta + x'_1)\} dx'_1 \]

References
Numerical Solution of Differential Equations in Power System Stability Analysis

The dynamics of a power system can be described by a set of first-order differential equations

\[ \dot{x} = f(x, y) \]  
(4.1)

and an algebraic set

\[ 0 = g(x, y) \]  
(4.2)

Since a solution of the above set of equations cannot be determined in analytical form, an approximate solution is generally sought using a numerical technique.

In general, methods of numerical integration employ a step-by-step process to determine a series of values for each dependent variable corresponding to a set of values of the independent variables. Integration methods fall into the main categories -- explicit or implicit, and single-step or multi-step. In explicit methods, the integration formulas are applied directly to each of the individual differential equations being solved. In implicit integration, the differential equations are algebraized and the resulting equations are solved simultaneously as a set. This is more complicated, but it has the reward of greater numerical stability. Single-step methods do not use information about the solution prior to the beginning of each integration step. Therefore, they are self-starting, which is convenient in the presence of discontinuities. Runge-Kutta is the most famous class of these methods. Multi-step methods require the storage of previous values of the variables and/or their derivatives, and thus, in principle more efficient. However, the process has to be restarted whenever a discontinuity occurs. Most multi-step methods use the open and/or closed difference formulas, of which the Adams family is best known.

Solution errors

There are several sources of numerical error in the solution over a given step. These are:

(a) Inexactness of the integration formulas used, i.e., truncation error.
(b) Computational inaccuracy due to arithmetic round off and, in methods that employ iteration, to non-exact convergence.
(c) Failure to achieve truly simultaneous solution in time of the set of equations (4.1) and (4.2), i.e., interface error.
(d) Approximation error, where certain assumptions about the behaviors of the variables or the linearity of the equations over a given step are made in return for computational economies.
(e) Failure to detect and apply limiting and limit back-off of variables at exactly the correct times, due to the use of finite time increments, and often to imprecise limit handling techniques.

With any solution method, truncation, interface, approximation, and limit errors can be kept within acceptable bounds by using sufficiently small step lengths. This tends to be detrimental to overall computing speed.
Stability of integration methods

The error in the solution at the end of any integration step is some function of the errors incurred as above during the step, and the error inherited from the beginning of the step. Numerical stability of the method is concerned with the propagation of error over many successive steps. An unstable method is one in which the error tends to accumulate so that it eventually "blows up" and swamps the true solution. The methods that have been most widely used in the power system problem (e.g., explicit Euler, Runge-Kutta) are not very stable ones, compared with the more recently used implicit methods.

The less stable an integration method is, the more it is necessary on a given problem to limit the generation and thus propagation of errors by using high-order (low truncation error) versions of the method, by converging iteration cycles accurately, by using high-precision arithmetic, and most of all, by using small step length. In the present power system problem, interface error is a special hazard which has been given much attention in the design of overall solution schemes. All these measures tend to increase the overall computing time. The difficulties are exacerbated if the problem itself is mathematically "stiff", as described below.

Problem stiffness

The problem is stiff if the ratio between the largest and smallest time constants is high. More precisely, stiffness is measured by the ratio between the largest and smallest eigenvalues of the linearized system.

On a stiff problem, a relatively unstable integration method will need very small step lengths to track accurately the rapidly changing components in the system response in order to maintain truncation (and other) errors at sufficiently low levels. This is the case even when these components are small magnitude fluctuations (quiescent modes) superimposed on slower varying responses, and which have very little effect on the solutions of the main variables of interest. On the other hand, a more stable integration method can tolerate much larger errors per step, because they are not going to be propagated as much. Hence, it is possible to use larger step lengths and/or be less concerned to minimize other errors for the same overall accuracy of solution.

The advantages of highly stable methods over weakly stable ones tend to reduce as the problems to be solved become less stiff. The classical "constant voltage behind transient reactance" stability model is not stiff at all, unless machine inertias vary widely. Stiffness increases with the detail of synchronous machine modeling. For instance, subtransient time constants are an order of magnitude smaller than transient time constants. Particular sources of stiffness are the small time constants that can be found in excitation control models. It is also important to recognize that stiffness is not simply identifiable from the physical time constants in the input data. There is hidden stiffness in the algebraic equations, especially with non-impedance loads.

Specific integration methods

Of the numerous integration methods found in the literature those that have found useful application to the power system stability problem are described below.

Euler's method

This least accurate low-stability method has been widely used in the past, because of its simple implementation. The basic application is as follows:
NUMERICAL SOLUTION OF THE TRANSIENT STABILITY PROBLEM

Starting at point \((x_{n-1}, y_{n-1})\), \(\dot{x}_{n-1}\) is computed from equation (4.1) and \(x_n\) is obtained as

\[ x_n = x_{n-1} + h \dot{x}_{n-1} \]  \(\text{(4.3)}\)

where \(h\) is the integration step length. Then equation (4.2) is solved to obtain \(y_n\).

Although there is no interface error, the scheme is inefficient. Euler's method demand very small step lengths unless the power system model is very simple and non-stiff. Thus, while only a single evaluation of \(x\) is made per step, the network has to be solved a very large number of times for the entire solution, which consumes perhaps eighty percent or more of the total computation time.

**The modified Euler method**

In the application of Euler's method, a value of \(\dot{x}_{n-1}\) computed at the beginning of the interval is assumed to apply over the entire interval. An improvement can be obtained by computing preliminary values of \(x\) and \(y\) as

\[ x_n^0 = x_{n-1} + h \dot{x}_{n-1} \]
\[ y_n^0 \]

and \(y_n^0\) from equation (4.2), and using these values \((x_n^0, y_n^0)\) in equation (4.1) to compute the approximate value of \(\dot{x}_n\) at the end of the interval, i.e.,

\[ \dot{x}_n^0 = f(x_n^0, y_n^0) \]

Then, an improved value \(x_n^1\) can be found using the average of \(\dot{x}_{n-1}\) and \(\dot{x}_n^0\) as follows:

\[ x_n^1 = x_{n-1} + \frac{\dot{x}_{n-1} + \dot{x}_n^0}{2} h \]  \(\text{(4.4)}\)

\(y_n^1\) is then computed from equation (4.2). Using \((x_n^1, y_n^1)\), a third approximation \(x_n^2\) can be obtained by the same process

\[ x_n^2 = x_{n-1} + \frac{\dot{x}_{n-1} + \dot{x}_n^1}{2} h \]  \(\text{(4.5)}\)

and a fourth

\[ x_n^3 = x_{n-1} + \frac{\dot{x}_{n-1} + \dot{x}_n^2}{2} h \]  \(\text{(4.6)}\)

This process can be continued until two consecutive estimates for \(x\) are within a specified tolerance. The entire process is then repeated for the next step.

**Runge-Kutta method**

A fourth-order version of the Runge-Kutta method can be written as:

\[ k_1 = h f(x_{n-1}, y_{n-1}) \]  \(\text{(4.7a)}\)
\[ k_2 = h f(x^1, y^1) \]  \(\text{(4.7b)}\)

where

\[ x^1 = x_{n-1} + k_1 / 2 \]
NUMERICAL SOLUTION OF THE TRANSIENT STABILITY PROBLEM

\[ k_3 = hf(x^2, y^2) \]  \hspace{1cm} (4.7c)

where
\[ x^2 = x_{n-1} + k_2 / 2 \]

\[ k_4 = hf(x^3, y^3) \]  \hspace{1cm} (4.7d)

where
\[ x^3 = x_{n-1} + k_3 \]

\[ x_n = x_{n-1} + (k_1 + 2k_2 + 2k_3 + k_4) / 6 \]  \hspace{1cm} (4.7e)

In the rigorous (time consuming) interfacing schemes, equation (4.2) is solved exactly prior to stages (4.7b) - (4.7d), to provide the values of \( y \) corresponding to those of \( x \).

An extrapolation of \( y \) avoids these intermediate solutions of equation (4.2). Having thus provided the values of \( y \) in (4.7b) - (4.7d), and then having obtained \( x_n \) from (4.7e), the network equation (4.2) is solved to give \( y_n \). At this point, reintegration with improved interpolated estimates of \( y \) in (4.7b) and (4.7c) and \( y^3 = y_n \) is performed where dictated by the interface-error control mechanism.

Predictor-Corrector methods

A predictor-corrector pair consists of an open formula and a closed formula in the multi-step category. The predictor is applied once per step to provide good initial condition for the corrector. The corrector itself may be applied with a fixed number of iterations per step, or it may be iterated to convergence.

A simple one-corrector iteration version for the power system problem is illustrated below:

(a) Compute \( \dot{x}_{n-1} = f(x_{n-1}, y_{n-1}) \) from equation (4.1)

(b) Predict \( x_n \) from
\[ x_n = x_{n-1} + h(23\dot{x}_{n-1} - 16\dot{x}_{n-2} + 5\dot{x}_{n-3}) / 12 \]  \hspace{1cm} (4.8a)

(c) Compute \( y_n \) from equation (4.2)

(d) Compute \( \dot{x}_n = f(x_n, y_n) \) from equation (4.1)

(e) Correct \( x_n \) from
\[ x_n = x_{n-1} + h(5f(x_n, y_n) + 8\dot{x}_{n-1} - \dot{x}_{n-2}) / 12 \]  \hspace{1cm} (4.8b)

(f) Compute \( y_n \) from equation (4.2)

The above implementation eliminates interface error if equation (4.2) is solved exactly in stages (c) and (f). The corrector may be iterated in a loop between stages (d) and (e), with \( y_n \) constant at the predicted (or extrapolated) value. Alternatively, stage (f) may be included in the loop, at the expense of extra network solutions. If in this case the corrector is converged accurately, interface error is eliminated even if stage (c) is omitted.

The formulas for the other versions of the method have the same structure and involve virtually the same computational effort. They differ only in the coefficients and the numbers of previous values required. The lowest order version is the modified Euler, which is self-starting.
Implicit multi-step integration

These more modern methods were designed to overcome the deficiencies of the predictor-corrector approach on stiff problems.

It is noted that equation (4.8b) is a set of simultaneous equations in $x_n$, with $y_n$ the only other unknown. In fact, all closed multi-step formulas of this type can be expressed in the general form

$$x_n = k h f(x_n, y_n) + c$$  (4.9)

where $c$ is the sum of weighted $x$ and $\dot{x}$ terms backwards from time $t_{n-1}$, and $k$ is a constant coefficient.

The simplest implicit integration scheme is the trapezoidal rule of integration, corresponding to the corrector formula

$$x_n = x_{n-1} + h (f(x_n, y_n) + \dot{x}_{n-1}) / 2$$  (4.10)

When this is expressed in the form of equation (4.9)

$$k = 1/2, \text{ and } c = x_{n-1} + (1/2) h \dot{x}_{n-1}$$

Equation (4.9) can be solved for $x_n$ by a method such as generalized Newton-Raphson. This solution directly replaces the corrector iterations of the last section. A predictor of the type (4.8a) is now only a somewhat arbitrary means of providing good starting values for the iterations and may be replaced by any other convenient method of extrapolating $x$. Starting values for $y_n$ are obtained by extrapolation.

In the absence of saturation, equation (4.9) is usually linear in $x_n$ and a direct solution, equivalent to a single Newton iteration, can be made. Let the differential equations (4.1) be linear and of the form

$$\dot{x} = Ax + By$$  (4.11)

It may be noted that the inclusion of some nonlinearities that are present in the equations poses no serious problem.

Substituting equation (4.11) into equation (4.10), and rearranging,

$$[I - (h/2)A] x_n = (h/2) B y_n + [I + (h/2) A] x_{n-1} + (h/2) B y_{n-1} = (h/2) B y_n + c$$  (4.12)

where

$$c = [I + (h/2) A] x_{n-1} + (h/2) B y_{n-1}$$

A matrix solution of equation (4.12) gives $x_n$ directly in terms of $y_n$. In one approach, equation (4.12) is solved alternately with an iteration in the solution of equation (4.2) for $y_n$, until the process has converged and interface error is eliminated.

Compared with the forward-substitution scheme for iterating the corrector in the previous section, the direct solution of equation (4.12) requires a little more work per step, and is more complicated to program. However, much larger steps can be taken on stiff problems, at a great overall saving in computation.

Solution of Multi-Machine Transient Stability Problem Using Classical Machine Model

Although any of the numerical methods of solving differential equations discussed in the previous section can be used in solving the multi-machine transient stability problem, the
Implicit integration scheme using the trapezoidal rule has certain advantages over other methods when detailed machine and control system representations are employed. This is due to the fact that the method is generally more stable and can handle problem stiffness very effectively. As a rule, programming this method is more complicated compared to the other methods. Since it appears that it has become the method of choice for solving transient stability problem, the solution of the transient stability problem will be illustrated using this method.

It should be emphasized that in the formulation of the transient stability problem using the classical model, the problem is not stiff at all, unless the machine inertias vary over a very wide range and the loads have special characteristics which might introduce stiffness to the problem. Therefore any of the numerical methods discussed earlier would be equally suitable (except, perhaps, the Euler's method in its original form, which would require excessively small integration step length). Programming complexity would be about the same in using any of the methods. While the programming complexity in using any of the explicit methods is more or less independent of the modeling details, the complexity in using the implicit methods goes up considerably as the modeling detail increases. However, the extra programming effort is more than compensated for by the saving in computing time that is realizable through the use of a much larger integration step length permitted by these methods.

**Single-machine-infinite bus system**

The solution of the transient stability problem will first be demonstrated using a single-machine-infinite bus system as depicted in Figure 4.1.

![Fig. 4.1 A single-machine-infinite bus system.](image)

The swing equation is

\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P_m - P_e
\]

(4.13)

where

\[
P_e = \frac{E_i E_s}{X} \sin \delta
\]

Equation (4.13) can be written as

\[
\frac{d\delta}{dt} = \omega - \omega_o
\]

(4.14)

\[
\frac{d\omega}{dt} = \frac{\omega_o}{2H} (P_m - P_e)
\]

(4.15)

(recall that \( \delta = \theta - \omega_o t \), \( \therefore \frac{d\delta}{dt} = \frac{d\theta}{dt} - \omega_o = \omega - \omega_o \))

Applying trapezoidal rule to equations (4.14) and (4.15)

\[
\delta_n = \delta_{n-1} + (\omega_{n-1} + \omega_n - 2\omega_o) h / 2
\]

(4.16)
NUMERICAL SOLUTION OF THE TRANSIENT STABILITY PROBLEM

and

\[
\omega_n = \omega_{n-1} + \frac{h}{2} \left[ \frac{\omega_o}{2H} (P_m - P_{e(n-1)}) + \frac{\omega_o}{2H} (P_m - P_{e_n}) \right] \tag{4.17}
\]

where \( h \) is the integration step length, \( h = t_n - t_{n-1} \).

Substituting equation (4.17) into (4.16), we have

\[
\delta_n = -\frac{h^2 \omega_o}{8H} P_{en} + a_{n-1} \tag{4.18}
\]

where

\[
a_{n-1} = \delta_{n-1} + h(\omega_{n-1} - \omega_o) + \frac{h^2 \omega_o}{8H} [2P_m - P_{e(n-1)}] \]

Substituting the expression for \( P_{en} \), equation (4.18) can be written as

\[
\delta_n = -\frac{h^2 \omega_o}{8H} E_i E_j \sin \delta_n + a_{n-1} \tag{4.19}
\]

Equation (4.19) can be solved directly for \( \delta_n \) by Newton's method. Alternatively, an iterative method can be employed. A value of \( \delta_n \) is predicted by using a predictor formula and then equation (4.19) can be used as a corrector formula. A predictor formula can be easily derived from equations (4.16) and (4.17) by assuming that the output power remains constant throughout the integration step interval and equal to the value at the beginning of the interval. Then,

\[
\delta_n = \delta_{n-1} + h(\omega_{n-1} - \omega_o) + \frac{h^2 \omega_o}{4H} [P_m - P_{e(n-1)}] \tag{4.20}
\]

Alternatively, an improved prediction can be obtained by assuming the output power at the middle of the interval as constant throughout the time interval. Then,

\[
\delta_n = 2\delta_{n-1} - \delta_{n-2} + \frac{h^2 \omega_o}{2H} [P_m - P_{e(n-1)}] \tag{4.21}
\]

**Multi-machine system**

First, consider the multi-machine case where the system has been reduced retaining only the machine internal buses as detailed in Chapter 3. The equation for the output power of each machine has been derived earlier and is given by

\[
P_{ei} = \sum_{k=1}^{N} E_i E_k (G_{ik} \cos \delta_{ik} + B_{ik} \sin \delta_{ik}) \quad i, 1, 2, \cdots N \tag{4.22}
\]

The swing equations for the machines are

\[
\frac{2H_i}{\omega_o} \frac{d^2 \delta_i}{dt^2} = P_{mi} - P_{ei} \quad i, 1, 2, \cdots N \tag{4.23}
\]

Applying trapezoidal rule to equation (4.23), we can write, as before,
NUMERICAL SOLUTION OF THE TRANSIENT STABILITY PROBLEM

\[ \omega^n_i = \omega^{n-1}_i + \frac{h}{2} \left[ \frac{\omega_o}{2H_i} (P_{mi} - P_{ei}^{n-1}) + \frac{\omega_o}{2H_i} (P_{mi} - P_{ei}^n) \right] \]

(4.24)

and

\[ \delta^n_i = -\frac{h^2 \omega_o}{8H_i} P_{ei}^n + a_i^{n-1} \]

(4.25)

where

\[ a_i^{n-1} = \delta_i^{n-1} + h (\omega_i^{n-1} - \omega_o) + \frac{h^2 \omega_o}{8H_i} (2P_{mi} - P_{ei}^{n-1}) \]

for \( i = 1, 2, \ldots N \)

Equations (4.25) and (4.22) can be solved for \( \delta_i^n \)'s by Newton's method. In the Newton's formulation, the following linear equation is solved for \( \Delta \delta \) to update \( \delta \),

\[ \Delta a = J \Delta \delta \]

(4.26)

where \( J \) is the Jacobian matrix.

For example, for a three machine system, the elements of \( \Delta a \) will be given by

\[ \Delta a_1 = a_1^{n-1} - \delta_1^n - \frac{h^2 \omega_o}{8H_1} \left[ E_1^2 G_{11} + E_1 E_2 B_{12} \sin \delta_{12} + E_1 E_3 B_{13} \sin \delta_{13} \right] \]

etc.

The elements of the Jacobian matrix are

\[ j_{11} = 1 + \frac{h^2 \omega_o}{8H_1} \left[ E_1 E_2 B_{12} \cos \delta_{12} + E_1 E_3 B_{13} \cos \delta_{13} \right] \]

\[ j_{12} = -\frac{h^2 \omega_o}{8H_1} E_1 E_2 B_{12} \cos \delta_{12} \]

etc.

\[ j_{21} = -\frac{h^2 \omega_o}{8H_2} E_1 E_2 B_{21} \cos \delta_{21} \]

\[ j_{22} = 1 + \frac{h^2 \omega_o}{8H_2} \left[ E_1 E_2 B_{21} \cos \delta_{21} + E_2 E_3 B_{23} \cos \delta_{23} \right] \]

etc.

In the above illustration, the transfer conductances have been neglected for simplicity.

Alternatively, an iterative solution can be employed. A value of \( \delta_i^n \) is predicted by using one of the predictor formulas given by equations (4.20) and (4.21), and equation (4.25) is used as a corrector formula, repeating the corrector formula as many times as necessary to achieve convergence.

The above multi-machine modeling where all nodes except the machine internal buses are eliminated, although convenient for programming, has several serious shortcomings. These are:
(a) Since all the non-generator buses are eliminated, the resulting admittance matrix is usually full. (In contrast, the original bus admittance matrix is usually extremely sparse.) In dealing with a large system containing many synchronous machines, the computation can become unnecessarily time consuming.

(b) In applying the network reduction technique, all the loads are converted into fixed impedances. While this is valid for certain types of loads, for other types of loads this assumption may be grossly erroneous. In order to be able to model the loads more accurately, the load buses must be retained in the network.

(c) Information on voltages at many of the non-generator buses may be required for computation of power flow on the transmission lines in order to simulate relay operation and line tripping during stability computation.

In order to achieve the above objectives and also to take advantage of the network sparsity, it is advisable to work with the network in the intact form. Portions of the system may be reduced and unwanted buses eliminated provided network sparsity is not significantly affected.

The system shown in Figure 4.2 will be used to illustrate the stability computation when non-generator buses including load buses are retained. In the five bus system shown, there are generators at buses numbered 1, 2 and 3 and loads at buses numbered 4 and 5. A procedure for working with the full network has been discussed in some detail in Chapter 3. Here the relevant steps will be repeated.

![Fig. 4.2 A five bus, three machine system.](image)

The bus currents and voltages are related by

\[ \mathbf{I} = \mathbf{YV} \]

where \( \mathbf{Y} \) is the network admittance matrix, \( \mathbf{V} \) is the vector of bus voltages and \( \mathbf{I} \) is the vector of injected currents.

The machines are represented by constant voltage magnitudes behind transient reactances. These can be replaced by current sources as illustrated in Figure 4.3. Therefore, the currents injected at buses 1, 2 and 3 would be \( E_1 / jx_{d1}' \), \( E_2 / jx_{d2}' \), and \( E_3 / jx_{d3}' \), respectively, where \( E_1 \), \( E_2 \), and \( E_3 \) are the machine internal voltages and \( x_{d1}' \), etc. are the transient reactances.
The network admittance matrix would be modified by including the shunt elements. For example, the element $Y_{11}$ would be modified to

$$Y'_{11} = Y_{11} + 1/jx_{d1}'$$

Similar modifications will apply to $Y_{22}$ and $Y_{33}$.

If loads are represented as constant impedances, these would be included in the network admittance matrix by modifying the corresponding elements of the matrix. The current injection at the load buses would then be zero.

For non-impedance loads, the corresponding current is computed using an estimate of the bus voltage which would be updated as improved estimates of the voltages are obtained during the iterative computation. In the interest of improved convergence, it is advisable to convert the major part of the non-impedance load into constant impedance and include it into the network admittance matrix. The injected current corresponding to the remaining portion of the load can then be adjusted to match the total load.

Equation (4.27) can now be solved by factoring the admittance matrix $Y$ into upper and lower triangular matrices, utilizing one of the sparsity oriented triangular factorization techniques, and then by forward and backward substitution. Since the matrix $Y$ remains constant unless there is a change in the network due to a disturbance, etc., it is triangularized only as many times as there are changes in the network during a stability computation.

The machine currents are computed from

$$I = (E - V)/jx_d'$$

and power from

$$P = \text{Re}(E^*I)$$

The procedure for computing the machine power outputs by the above method and updating by using the corrector formula of equation (4.25) corresponds to the iterative procedure described earlier. As before, the solution can also be obtained directly by Newton's method.

**Handling of constant current load in network solution during stability computation using Newton's method**

The two machine-three bus system shown in Figure 4.4 will be used to illustrate the handling of constant current load in stability computation using Newton's method. As explained in Chapter 7, representation of load by constant current (or power) in stability studies is not recommended. However, since most commercially available stability programs allow for constant current load
and loads are frequently represented by constant current in stability studies, a method of handling constant current load is presented here.

In the system shown in Figure 4.4, buses 1 and 2 are generator buses and bus 3 is a load bus having a constant current load. The phase relationship of the voltage and current at bus 3 is shown in Figure 4.5.

We have

\[ i_1 = I \cos \phi = \text{constant} \]
\[ i_2 = I \sin \phi = \text{constant} \]

From the phasor diagram,

\[ r = i_1 \cos \theta + i_2 \sin \theta \]
\[ s = i_1 \sin \theta - i_2 \cos \theta \]

\[ \cos \theta = \frac{e}{\sqrt{e^2 + f^2}} = \frac{e}{|V|}, \quad \sin \theta = \frac{f}{\sqrt{e^2 + f^2}} = \frac{f}{|V|} \]

\[ r = \frac{i_1 e + i_2 f}{\sqrt{e^2 + f^2}} \]

and

\[ s = \frac{i_1 f - i_2 e}{\sqrt{e^2 + f^2}} \]
We can compute the partial derivatives as
\[ \frac{\partial r}{\partial e} = \frac{i_1}{|V|} + (i_1 e + i_2 f) \frac{-e}{|V|^3}, \]
\[ \frac{\partial r}{\partial f} = \frac{i_2}{|V|} + (i_1 e + i_2 f) \frac{-f}{|V|^3}, \]
\[ \frac{\partial s}{\partial e} = \frac{-i_2}{|V|} + (i_2 e - i_1 f) \frac{e}{|V|^3}, \]
\[ \frac{\partial s}{\partial f} = \frac{i_1}{|V|} + (i_2 e - i_1 f) \frac{f}{|V|^3}, \]  
\begin{equation}
(4.32)
\end{equation}

If the reactive part of the load is constant impedance, then
\[ i_2 = -|V|B \]
where \( B \) is the load susceptance \((B = -1/X)\)

\[ : \quad r = \frac{i_1 e}{\sqrt{e^2 + f^2}} - f B \]
and
\[ s = \frac{i_1 f}{\sqrt{e^2 + f^2}} + e B \]

The partial derivatives are then
\[ \frac{\partial r}{\partial e} = \frac{i_1}{|V|} - \frac{i_1 e^2}{|V|^3}, \quad \frac{\partial r}{\partial f} = -B - \frac{i_1 e f}{|V|^3} \]
\[ \frac{\partial s}{\partial e} = B - \frac{i_1 e f}{|V|^3}, \quad \frac{\partial s}{\partial f} = \frac{i_1}{|V|} - \frac{i_1 f^2}{|V|^3} \]

If both real and reactive parts are constant impedances then
\[ i_1 = |V|G, \quad i_2 = -|V|B \]
where \( G \) is the load conductance.

\[ : \quad r = e G - f B, \quad s = f G + e B \]
and
\[ \frac{\partial r}{\partial e} = G, \quad \frac{\partial r}{\partial f} = -B, \quad \frac{\partial s}{\partial e} = B, \quad \frac{\partial s}{\partial f} = G \]

For the network shown in Figure 4.4, after changing the constant voltage sources into current sources,
\[ I_1 = r_1 + js_1 = E_1 \angle \delta_1 / (jx'_{d1}) \]
\[ = (E_1 / x'_{d1}) \sin \delta_1 - j (E_1 / x'_{d1}) \cos \delta_1 \]
NUMERICAL SOLUTION OF THE TRANSIENT STABILITY PROBLEM

\[ I_2 = r_2 + js_2 = E_2 \angle \delta_2 / (j \omega_d) \]
\[ = (E_2 / x'_d) \sin \delta_2 - j (E_2 / x'_d) \cos \delta_2 \]

and, assuming a constant current load at bus 3,

\[ I_3 = -r_3 - js_3 = -(i_1 e_3 + i_2 f_3) / \|V_3\| - j (i_1 f_3 - i_2 e_3) / \|V_3\| \]

Also

\[ I_1 = r_1 + js_1 = [G_{11} + j (B_{11} - 1 / x'_d)] (e_1 + j f_1) + 
(G_{12} + j B_{12}) (e_2 + j f_2) + (G_{13} + j B_{13}) (e_3 + j f_3) \]

\[ I_2 = r_2 + js_2 = [G_{21} + j (B_{21} - 1 / x'_d)] (e_1 + j f_1) + 
(G_{22} + j B_{22}) (e_2 + j f_2) + (G_{23} + j B_{23}) (e_3 + j f_3) \]

\[ I_3 = -r_3 - js_3 = (G_{31} + j B_{31}) (e_1 + j f_1) + 
(G_{32} + j B_{32}) (e_2 + j f_2) + (G_{33} + j B_{33}) (e_3 + j f_3) \]

The above equations can be put into Newton's formulation, after separating the real and imaginary parts.

\[
\begin{bmatrix}
\Delta r_1 \\
\Delta s_1 \\
\Delta r_2 \\
\Delta s_2 \\
\Delta r_3 \\
\Delta s_3 \\
\end{bmatrix} = \begin{bmatrix}
G_{11} & - (B_{11} - 1 / x'_d) & G_{12} & - B_{12} & G_{13} & - B_{13} \\
(B_{11} - 1 / x'_d) & G_{11} & B_{12} & G_{12} & B_{13} & G_{13} \\
G_{21} & - B_{21} & G_{22} & - (B_{22} - 1 / x'_d) & G_{23} & - B_{23} \\
B_{21} & G_{21} & (B_{22} - 1 / x'_d) & G_{22} & B_{23} & G_{23} \\
G_{31} & - B_{31} & G_{32} & - B_{32} & G'_{33} & - B'_{33} \\
B_{31} & G_{31} & B_{32} & G_{32} & B_{33} & G_{33} \\
\end{bmatrix} \begin{bmatrix}
\Delta e_1 \\
\Delta f_1 \\
\Delta e_2 \\
\Delta f_2 \\
\Delta e_3 \\
\Delta f_3 \\
\end{bmatrix}
\]

where

\[ G'_{33} = G_{33} + i_1 / \|V_3\| - (i_1 e_3 + i_2 f_3) e_3 / \|V_3\|^3 \]
\[ B'_{33} = B_{33} - i_2 / \|V_3\| + (i_1 e_3 + i_2 f_3) f_3 / \|V_3\|^3 \]
\[ B'_{33} = B_{33} - i_2 / \|V_3\| + (i_2 e_3 - i_1 f_3) f_3 / \|V_3\|^3 \]
\[ G_{33}^* = G_{33} + i_1 / \|V_3\|^3 + (i_2 e_3 - i_1 f_3) f_3 / \|V_3\|^3 \]

where \( i_1 \) and \( i_2 \) are the real and reactive components of the constant current load.

\( \Delta r_1, \Delta s_1, \) etc. are computed in the usual way. \( \Delta e_1, \Delta f_1, \) etc. are obtained by solving equation (4.36).
Solution of Faulted Networks

In normal operations power systems are balanced or very nearly so. This means that the three phases are symmetrical. Therefore, for computational purposes, power systems may be represented on a single-phase line-to-neutral basis. All the computation procedures presented earlier are based on the assumption of balanced, symmetrical operations. During a three-phase fault a power system remains symmetrical and therefore, the single-phase representation remains valid. Simulation of the fault is accomplished by suitably adjusting the network bus admittance matrix. For example, if the fault is a three-phase bus-to-ground short circuit, it is simulated simply by deleting the row and column corresponding to the faulted bus. If the bus fault is through an impedance it is simulated by adding the fault admittance to the diagonal term corresponding to the faulted bus. For a fault somewhere along a transmission line, the diagonal and off-diagonal elements corresponding to the end buses of the faulted line are appropriately modified. Following the change in the bus admittance matrix that reflects the fault, stability computations can proceed as described earlier.

During faults other than three-phase, such as line-to-ground, line-to-line or open conductors, power systems are unsymmetrical, and the single-phase representation is not sufficient. In such cases the method of symmetrical components is generally used.

In the method of symmetrical components an unsymmetrical set of voltage or current phasors is resolved into symmetrical sets of components. The unbalanced three-phase system is resolved into three balanced (symmetrical) systems of phasors called the positive, negative and zero sequence components. The positive-sequence components have the same phase sequence as the original phasors. They are equal in magnitude and displaced from each other by 120°. The negative-sequence components, equal in magnitude and displaced from each other by 120° in phase, have a phase sequence opposite to that of the original phasors. The zero-sequence components are equal in magnitude and have zero phase displacement.

Denoting the three phases of the original system by $a$, $b$ and $c$, and the symmetrical components by 1, 2 and 0, the transformation can be expressed in matrix form, for example, for the voltage phasors, as

\[
\begin{bmatrix}
V_a \\
V_b \\
V_c
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & a^2 & a \\
1 & a & a^2
\end{bmatrix}
\begin{bmatrix}
V_{a0} \\
V_{a1} \\
V_{a2}
\end{bmatrix}
\]

or

\[
V_{abc} = TV_{012}
\]  
(4.37)

In the above equation the $a$’s are the operators defined by

\[
a = 1\angle 120°, \quad a^2 = 1\angle -120°, \quad a^3 = 1\angle 0°
\]

so that

\[
V_{b1} = a^2 V_{a1}, \quad V_{c1} = a V_{a1}
\]

and

\[
V_{b2} = a V_{a2}, \quad V_{c2} = a^2 V_{a2}
\]

Also

\[
V_{a0} = V_{b0} = V_{c0}
\]
The reverse transformation is

$$V_{012} = T^{-1}V_{abc}$$ \hspace{1cm} (4.38)

where

$$T^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

Similar equations also apply to currents, i.e.,

$$I_{abc} = TI_{012}$$ \hspace{1cm} (4.39)

$$I_{012} = T^{-1}I_{abc}$$ \hspace{1cm} (4.40)

The total complex power in the three phases

$$P + jQ = V'_{abc}^*I_{abc} = V'_{012}^*T'T_{012}^* = 3V'_{012}I_{012}^*$$ \hspace{1cm} (4.41)

since $a$ and $a^2$ are complex conjugates

Equation (4.41) shows how complex power can be computed from the symmetrical components of the voltages and currents in an unbalanced three-phase circuit. In per unit, equation (4.41) reduces to

$$P + jQ = V'_{012}I_{012}^*$$ \hspace{1cm} (4.42)

In a balanced three-phase circuit, the three sequences are independent. This means that currents of each phase sequence will produce voltage drops which are of the same sequence and are independent of currents of other sequences. In a balanced system, currents of any one sequence may be considered to flow in an independent network composed of the impedances to the current of that sequence only. The single-phase equivalent circuit composed of the impedances to current of any one sequence only is called the sequence network for that particular sequence. It includes any generated emfs of that sequence.

In order to show the independence of the three sequences consider a balanced static three-phase circuit with equal self and mutual impedances. The voltage-current relationship can be written in matrix form as

$$V_{abc} = Z_{abc}I_{abc}$$ \hspace{1cm} (4.43)

with $Z_{aa} = Z_{bb} = Z_{cc}$ and $Z_{ab} = Z_{ba} = Z_{ac} = Z_{ca}$, etc.

Applying the transformation (4.37) to (4.43)

$$V_{012} = Z_{012}I_{012}$$ \hspace{1cm} (4.44)

where

$$Z_{012} = T^{-1}Z_{abc}T = \text{diag}[Z_0 \ Z_1 \ Z_2]$$

$$Z_0 = Z_{aa} + 2Z_{ab}, \quad Z_1 = Z_2 = Z_{aa} - Z_{ab}$$
\[ \therefore V_{a0} = Z_0 I_{a0}, \quad V_{a1} = Z_1 I_{a1}, \quad V_{a2} = Z_2 I_{a2} \]

\( Z_0, Z_1, Z_2 \) are called the zero-, positive-, and negative-sequence impedances, respectively.

The positive- and negative-sequence impedances of any balanced static three-phase circuit are equal. For rotating machinery, however, the positive-sequence impedance usually differs from the negative-sequence impedance because of the interaction between the stator and rotor windings. The zero-sequence impedance depends on the sign of the mutual impedance in the original circuit. If the self and mutual impedances are of the same sign, as in the case of a transmission line or cable, the zero-sequence impedance is higher than the positive- or negative-sequence impedance. In a three-phase machine, the mutual impedance is of opposite sign from the self impedance, and the zero-sequence impedance is lower than the positive- or negative-sequence impedance.

Since the generated voltages are of positive-sequence, the generated power of a synchronous machine and the synchronizing power between the various synchronous machines of a power system are positive-sequence power. Therefore, the positive-sequence network is of primary interest in a stability study.

### Analysis of Unsymmetrical Faults

Unsymmetrical faults occur as single line-to-ground faults, line-to-line faults, double line-to-ground faults, or one or two open conductors. The path of the fault current from line-to-line or line-to-ground may or may not contain impedance.

For the purpose of analyzing fault at any point in a power system, the system can be replaced by the Thevenin's voltage in series with a Thevenin's impedance. The phase voltage and current relationships at the point of fault can then be written in matrix form as

\[ V_{abc} = E_{abc} - Z_{abc} I_{abc} \quad (4.45) \]

where

- \( V_{abc} \) = vector of phase voltages at point of fault
- \( E_{abc} \) = vector of Thevenin or system internal voltage
- \( I_{abc} \) = vector of phase currents flowing out of the system and into the fault
- \( Z_{abc} \) = matrix of the Thevenin impedances, assumed symmetric

In terms of the symmetrical components, equation (4.45) becomes

\[ V_{012} = E_{012} - Z_{012} I_{012} \quad (4.46) \]

In expanded form, equation (4.46) can be written as

\[
\begin{bmatrix}
V_{a0} \\
V_{a1} \\
V_{a2}
\end{bmatrix} =
\begin{bmatrix}
0 \\
E_a \\
0
\end{bmatrix} -
\begin{bmatrix}
Z_0 & Z_1 & Z_2
\end{bmatrix}
\begin{bmatrix}
I_{a0} \\
I_{a1} \\
I_{a2}
\end{bmatrix}
\]

\( Z_0, Z_1, Z_2 \) are the zero-, negative- and positive-sequence Thevenin impedances of the respective sequence networks, measured between point of fault and the reference bus. Note that since the generated voltages are assumed to be balanced before the fault, \( E_{a0} = E_{a2} = 0 \), and \( E_{a1} = E_a \).
Each type of fault can be analyzed using equation (4.47) together with equations that describe conditions at the fault.

**Single line-to-ground fault**
For a single line-to-ground fault, assuming the fault is in phase \(a\),
\[
I_b = I_c = 0, \quad V_a = 0
\]
Therefore, from (4.37) and (4.40),
\[
V_{a0} + V_{a1} + V_{a2} = 0
\]
\[
I_{a0} = I_{a1} = I_{a2} = I_a / 3
\]
which yield, from equation (4.47),
\[
I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_0}
\] (4.48)

**Line-to-line fault**
For a line-to-line fault, assuming the fault is between phases \(b\) and \(c\),
\[
V_b = V_c, \quad I_a = 0, \quad I_b = -I_c
\]
Therefore, from (4.38) and (4.40),
\[
I_{a0} = 0, \quad I_{a2} = -I_{a1}, \quad V_{a1} = V_{a2}
\]
and from (4.47), \(V_{a0} = 0\) (with finite \(Z_0\))
Therefore, from (4.47),
\[
I_{a1} = \frac{E_a}{Z_1 + Z_2}
\] (4.49)

**Double line-to-ground fault**
For a double line-to-ground fault
\[
V_b = V_c = 0, \quad I_a = 0
\]
Therefore, from (4.38),
\[
V_{a1} = V_{a2} = V_{a0}
\]
and from (4.47),
\[
I_{a1} = \frac{E_a}{Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}}
\] (4.50)

If the line-to-ground or double line-to-ground fault is through an impedance \(Z_f\), it can be easily verified that equations (4.48) and (4.50) apply after modifying the zero sequence impedance to \(Z_0 + 3Z_f\).
Equations (4.48) to (4.50) suggest that an unbalanced fault can be represented by connecting a shunt impedance $z_f$ at the point of fault of the original (positive-sequence) network as shown in Figure 4.6. The network admittance matrix would be adjusted accordingly. The value of $z_f$ would depend on the type of fault as listed in Table 4.1. In Table 4.1, $Z_0$ and $Z_2$ are the equivalent zero- and negative-sequence impedance as viewed from the point of fault.

Table 4.1

<table>
<thead>
<tr>
<th>Type of fault</th>
<th>$z_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\phi$</td>
<td>0</td>
</tr>
<tr>
<td>L-G</td>
<td>$Z_2 + Z_0$</td>
</tr>
<tr>
<td>L-L</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>L-L-G</td>
<td>$Z_2Z_0/(Z_2 + Z_0)$</td>
</tr>
</tbody>
</table>

One open conductor

Figure 4.7 shows the section of a transmission line with an open conductor (phase $a$). The voltages on the two sides of the opening are as shown.

$$
\begin{align*}
V_a & = I_a' \\
V_b & = I_b' \\
V_c & = I_c'
\end{align*}
$$

Fig. 4.7 Section of a transmission line showing an open conductor.

From Figure 4.7 we have

$$
I_a = 0, V_b' = 0, V_c' = 0
$$

The voltage and current relationships in terms of symmetrical components would be given by equation (4.47), except that $Z_0, Z_1, Z_2$ are the zero-, positive-, and negative-sequence impedances looking into the network at the point of open conductor. (Note that these impedances are
distinctly different from the sequence impedances that would be used for unbalanced short circuit at the same point.)

The component voltages and currents at the point of open conductor can therefore be expressed as

\[
V'_{a0} = V'_{a1} = V'_{a2} = V'_{aa} / 3
\]

\[
I_{a0} + I_{a1} + I_{a2} = 0
\]

and

\[
I_{a1} = \frac{E_a}{Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}}
\]

Therefore, the simulation of an open conductor in a transmission line is accomplished by connecting the parallel combination of the zero- and negative-sequence impedances looking into the network at the point of open conductor in series with the transmission line.

References


Synchronous machines play a vital role in power system stability, and a knowledge of the theory and performance of these machines is essential for a proper understanding of the subject. So far, the simplest synchronous machine representation -- the so-called classical model -- has been used to explain some of the fundamentals of power system stability. In this chapter we will review the basic theory of synchronous machines and derive models suitable for a more detailed analysis of the stability problem, including the effects of control.

A synchronous machine consists of two major components, the stator and the rotor, that are in relative motion and that are different in structure. In the usual machine the exciting magnetic field is produced by a set of coils (the field winding) on the rotor which rotates within the stator which supports and provides a magnetic flux path for the armature windings. There are ordinarily three armature coils placed around the stator at 120° electrical degrees apart so that, with uniform rotation of the magnetic field, voltages displaced 120° in phase will be produced in the coils. The rotor magnetic paths and all of its electrical circuits are assumed to be symmetrical about both the pole (direct) and interpole (quadrature) axes. The field winding is separate from the other rotor circuits and has its axis in line with the direct axis. The other rotor circuits are formed by the amortisseur or damper bars set in pole-face slots, arranged symmetrically and connected together at the ends in the case of salient-pole machines, and by the eddy current paths in the iron in the case of the solid rotors of the turbo-generators. The effect of these additional rotor circuits can be approximated by two sets of equivalent damper or amortisseur circuits with their axes in line with the direct and quadrature axes, respectively. Usually one circuit on each axis is sufficient to account for the various transient and subtransient effects. The symmetrical choice of the rotor circuits has the virtue of making all mutual inductances and resistances between direct and quadrature axis rotor circuits equal to zero.

All mutual inductances between stator and rotor circuits are periodic functions of rotor angular position. In addition, because of the rotor saliency, the self inductances of the stator phases and the mutual inductances between any two stator phases are also periodic functions of rotor angular position. Therefore, the resulting circuit equations are awkward to handle. However, if certain reasonable assumptions are made, a relatively simple transformation of variables will eliminate the troublesome functions of angle from the equations.

The assumptions are:

1. The windings are sinusoidally distributed along the air gap as far as all mutual effects with the rotor are concerned.
2. The stator slots cause no appreciable variation of any of the rotor inductances with change in rotor position.
3. Saturation may be neglected. The effect of saturation can be accounted for separately.

A schematic representation of a synchronous machine is shown in Figure 5.1.
The electrical performance of a synchronous machine may now be described by the following equations.

**Voltage Relations**

The voltage relationship of any of the armature windings is of the form

\[ e = \frac{d\psi}{dt} - ri \]  

(5.1)

where \( e \), \( \psi \) and \( i \) are the terminal voltage, total flux linkage, and current, of the winding, respectively, and \( r \) is the winding resistance. The direction of positive current corresponds to generator action.

Denoting the three phases by subscripts \( a, b, c \), the armature voltage equations can be written in matrix form as

\[
\begin{bmatrix}
  e_a \\
  e_b \\
  e_c
\end{bmatrix} = \frac{d}{dt} \begin{bmatrix}
  \psi_a \\
  \psi_b \\
  \psi_c
\end{bmatrix} - r \begin{bmatrix}
  i_a \\
  i_b \\
  i_c
\end{bmatrix}
\]  

(5.2)

or

\[ e_{abc} = \frac{d}{dt} \Psi_{abc} - r i_{abc} \]  

(5.3)

where \( e_{abc} \), \( \Psi_{abc} \), and \( i_{abc} \) are the vectors of the voltages, flux linkages and currents, respectively, of phases \( a, b \) and \( c \). The three phases, \( a, b, c \), are lettered in the direction of rotation of the rotor, as shown in Figure 5.1.
In a similar manner, the voltage relationships of the field and amortisseur windings, assuming an unspecified number of amortisseur windings, can be written in matrix form as

\[
\begin{bmatrix}
 e_{fd} \\
 0 \\
 \vdots \\
 0 \\
 \vdots \\
 \vdots \\
 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix}
 \psi_{fd} \\
 \psi_{ld} \\
 \vdots \\
 \psi_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix} + \begin{bmatrix}
 r_{fd} \\
 r_{ld} \\
 \vdots \\
 r_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix} \begin{bmatrix}
 i_{fd} \\
 i_{ld} \\
 \vdots \\
 i_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix}
\]

or

\[
V_r = \frac{d}{dt} \Psi_r + R_r i_r
\]

where

\[
V_r = \begin{bmatrix}
 e_{fd} \\
 0 \\
 \vdots \\
 0 \\
 \vdots \\
 \vdots \\
 \end{bmatrix}, \quad \Psi_r = \begin{bmatrix}
 \psi_{fd} \\
 \psi_{ld} \\
 \vdots \\
 \psi_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix}, \quad i_r = \begin{bmatrix}
 i_{fd} \\
 i_{ld} \\
 \vdots \\
 i_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix}
\]

and

\[
R_r = \text{diag}\left[ r_{fd}, r_{ld}, \ldots, r_{iq} \right]
\]

**Flux-Linkage Relations**

The armature flux-linkage relationships can be written in matrix form as

\[
\begin{bmatrix}
 \psi_a \\
 \psi_b \\
 \psi_c \\
 \end{bmatrix} = \begin{bmatrix}
 l_{aa} & l_{ab} & l_{ac} \\
 l_{ba} & l_{bb} & l_{bc} \\
 l_{ca} & l_{cb} & l_{cc} \\
 \end{bmatrix} \begin{bmatrix}
 i_a \\
 i_b \\
 i_c \\
 \end{bmatrix} + \begin{bmatrix}
 l_{a1d} & \cdots & l_{a1q} & \cdots \\
 \vdots & \ddots \vdots \ddots \ddots \vdots \ddots \\
 l_{a1d} & \cdots & l_{a1q} & \cdots \\
 \end{bmatrix} \begin{bmatrix}
 i_{fd} \\
 i_{ld} \\
 \vdots \\
 i_{iq} \\
 \vdots \\
 \vdots \\
 \end{bmatrix} \]

or

\[
\Psi_{abc} = L_{abc} i_{abc} + L_{a} i_r
\]

Similarly, the rotor flux-linkage relationships can be written as
Inductance Relations

Armature self-inductances

The current in phase \( a \) produces an mmf \( N_i a \), where \( N \) is the effective number of turns of the phase \( a \) winding. This mmf has a component \( N_i a \cos \theta \) along the direct axis, and a component \( -N_i a \sin \theta \) along the quadrature axis, where \( \theta \) is the angle of the direct axis from the axis of phase \( a \), measured in the direction of rotor rotation (Fig. 5.1). If \( P_d \) and \( P_q \) are the permeances of the magnetic circuits in the \( d \) and \( q \) axes, respectively, the fundamental flux components along these two axes are:

\[
\Phi_d = P_d N_i a \cos \theta, \quad \Phi_q = -P_q N_i a \sin \theta
\]

The components of these fluxes along the axis of phase \( a \) are \( \Phi_d \cos \theta \) and \( -\Phi_q \sin \theta \), respectively, and hence the flux which links phase \( a \) is

\[
P_d N^2 i_a \cos^2 \theta + P_q N^2 i_a \sin^2 \theta
\]

There is also some flux linking phase \( a \) that does not link the rotor -- the leakage flux. If the flux leakage paths have a permeance \( P_l \), assumed to be independent of \( \theta \), the leakage flux is \( P_l N_i a \) and the flux linkage due to it is

\[
N^2 P_l i_a = N^2 (P_d \cos^2 \theta + P_q \sin^2 \theta) i_a.
\]

The total flux linkage is therefore

\[
\Phi_a = N^2 (P_d + P_l) i_a \cos^2 \theta + N^2 (P_q + P_l) i_a \sin^2 \theta
\]

\[
= N^2 \left( \frac{P_d + P_l}{2} + \frac{P_q - P_l}{2} \cos 2\theta \right) i_a
\]

Since inductance is defined as flux linkage per ampere, the self-inductance of phase \( a \) may be expressed as

\[
L_{aa} = L_{aa0} + L_{aa2} \cos 2\theta \tag{5.10}
\]

Self-inductances of phases \( b \) and \( c \) are obtained by noting that they are displaced by \( \pm 120^\circ \) from phase \( a \). Therefore the self-inductances of phases \( b \) and \( c \) are:

\[
L_{bb} = L_{bb0} + L_{bb2} \cos 2(\theta - 120^\circ)
\]

\[
L_{cc} = L_{cc0} + L_{cc2} \cos 2(\theta + 120^\circ) \tag{5.11}
\]
Armature mutual inductances
The mutual inductances can be found in a similar manner. The flux produced by phase \(a\) that links phase \(b\), assuming the effective number of turns of phases \(a\), \(b\) and \(c\) are equal, is

\[
\phi_{ab} = \phi_d N \cos(\theta - 120^\circ) - \phi_q N \sin(\theta - 120^\circ)
\]

\[
= P_d N^2 i_a \cos \theta \cos(\theta - 120^\circ) + P_q N^2 i_a \sin \theta \sin(\theta - 120^\circ)
\]

\[
= N^2 \left[ -\frac{P_d + P_q}{4} + \frac{P_d - P_q}{2} \cos(2\theta - 120^\circ) \right] i_a
\]

The mutual inductances between phase \(a\) and phase \(b\) can thus be written in the form

\[
l_{ab} = l_{ba} = -L_{ab0} + L_{aa2} \cos(2\theta - 120^\circ) \quad (5.12)
\]

Note that the magnitudes of the variable parts of the self and mutual inductances are the same and that the magnitude of the constant part of the mutual inductance is half that of the constant part of the self inductance, excluding leakage.

Similarly,

\[
l_{bc} = l_{cb} = -L_{cb0} + L_{aa2} \cos 2\theta
\]

\[
l_{ca} = l_{ac} = -L_{ab0} + L_{aa2} \cos(2\theta + 120^\circ)
\]

Rotor self-inductances
Since the effects of stator slots and of saturation are being neglected, all the rotor self-inductances \(l_{fdd}, l_{11d}, l_{11q}, \text{etc.}\) are constants.

Rotor mutual inductances
All mutual inductances between any two circuits both in the direct axis and between any two circuits both in the quadrature axis are constants and also, \(l_{j1d} = l_{j1d}, \text{etc.}\). Because of the rotor symmetry there is no mutual inductance between any direct and any quadrature axis circuit.

Mutual inductances between stator and rotor circuits
From consideration of the variable coupling between rotor circuits and armature circuits as function of rotor position, it follows that all stator-rotor mutual inductances vary sinusoidally with angle and that they are maximum when the two coils in question are in line. Thus

\[
l_{a_fd} = l_{f_ad} = L_{a_fd} \cos \theta
\]

\[
l_{b_fd} = l_{f_bd} = L_{a_fd} \cos(\theta - 120^\circ)
\]

\[
l_{c_fd} = l_{f_cd} = L_{a_fd} \cos(\theta + 120^\circ)
\]

\[
l_{a_1d} = l_{1_ad} = L_{a_1d} \cos \theta
\]

\[
l_{b_1d} = l_{1_bd} = L_{a_1d} \cos(\theta - 120^\circ)
\]

\[
l_{c_1d} = l_{1_cd} = L_{a_1d} \cos(\theta + 120^\circ)
\]

\[
\text{etc.}
\]
SYNCHRONOUS MACHINES

\[ I_{a1q} = I_{iaq} = -L_{a1q} \sin \theta \]
\[ I_{b1q} = I_{ibq} = -L_{a1q} \sin(\theta - 120^\circ) \]
\[ I_{c1q} = I_{icq} = -L_{a1q} \sin(\theta + 120^\circ) \]

etc.

Transformation of Equations

The equations of the synchronous machine can be greatly simplified by applying a certain transformation of variables. Referring to Figure 5.1, if the mmfs produced by the three armature currents are resolved along the direct and quadrature axes, currents proportional to the resultant mmfs in the direct and quadrature axes can be defined as

\[
\begin{align*}
I_d &= \frac{2}{3} \left[ I_a \cos \theta + I_b \cos(\theta - 120^\circ) + I_c \cos(\theta + 120^\circ) \right] \\
I_q &= -\frac{2}{3} \left[ I_a \sin \theta + I_b \sin(\theta - 120^\circ) + I_c \sin(\theta + 120^\circ) \right]
\end{align*}
\]

This is the basis of the transformation originally used by R. H. Park. In defining the above transformation, the axis of phase \(a\) was chosen as reference for convenience. Note that the effect of the transformation is to transform the phase quantities into variables in a reference frame which is fixed on the rotor.

The factor 2/3 is introduced, so that, for balanced phase currents of any given (maximum) magnitude, the maximum values of \(I_d\) and \(I_q\) as the phase of the currents is varied will be of the same magnitude.

Since three phase currents -- \(i_a, i_b, i_c\) -- are to be transformed, three new variables are required in order for the transformation to be reversible. A third (stationary) current, \(i_o\), which is proportional to the conventional zero sequence current of symmetrical component theory, is therefore introduced.

The transformation can therefore be expressed in matrix form as

\[ i_{dqo} = Ti_{abc} \]  \hspace{1cm} (5.15)

where

\[ i_{dqo} = \begin{bmatrix} I_d \\ I_q \\ I_o \end{bmatrix} \]

and

\[ T = \frac{2}{3} \begin{bmatrix} \cos \theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\
-\sin \theta & -\sin(\theta - 120^\circ) & -\sin(\theta + 120^\circ) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \]  \hspace{1cm} (5.16)

The inverse transformation is

\[ i_{abc} = T^{-1} i_{dqo} \]  \hspace{1cm} (5.17)
where

\[
T^{-1} = \begin{bmatrix}
\cos \theta & -\sin \theta & 1 \\
\cos(\theta - 120^\circ) & -\sin(\theta - 120^\circ) & 1 \\
\cos(\theta + 120^\circ) & -\sin(\theta + 120^\circ) & 1
\end{bmatrix}
\]  

(5.18)

Similar transformation can also be defined for armature flux linkages and voltages.

\[
\Psi_{dqo} = T \Psi_{abc}
\]  

(5.19)

\[
e_{dqo} = T e_{abc}
\]  

(5.20)

where

\[
\Psi_{dqo} = \begin{bmatrix}
\psi_d \\
\psi_q \\
\psi_0
\end{bmatrix}
\quad \text{and} \quad
e_{dqo} = \begin{bmatrix}
e_d \\
e_q \\
e_o
\end{bmatrix}
\]

Applying the transformation (5.17) to equation (5.9), we obtain

\[
\Psi_r = l_{rs} T^{-1} i_{dqo} + l_{rr} i_r
\]

Using the relationships (5.13), the above reduces to

\[
\Psi_r = L_{rs} i_{dqo} + L_{rr} i_r
\]  

(5.21)

where

\[
L_{rs} = \begin{bmatrix}
-\frac{3}{2} L_{qfd} & 0 & 0 \\
-\frac{3}{2} L_{a1d} & 0 & 0 \\
0 & -\frac{3}{2} L_{a1q} & 0
\end{bmatrix}
\]

and

\[
L_{rr} = l_{rr}
\]

Similarly, equation (5.7) can be transformed as

\[
T^{-1} \Psi_{dqo} = l_{ss} T^{-1} i_{dqo} + l_{sr} i_r
\]

or

\[
\Psi_{dqo} = T l_{ss} T^{-1} i_{dqo} + T l_{sr} i_r
\]  

(5.22)

After carrying out the indicated matrix operations, using the relationships (5.10) - (5.13), the above reduces to

\[
\Psi_{dqo} = L_{ss} i_{dqo} + L_{sr} i_r
\]  

(5.23)

where

\[
L_{ss} = \text{diag}[-L_d, -L_q, -L_o]
\]
Synchronous Machines

\[ L_d = L_{a0} + L_{a0} + \frac{3}{2} L_{aa2} \]
\[ L_q = L_{a0} + L_{a0} - \frac{3}{2} L_{aa2} \]
\[ L_o = L_{a0} - 2 L_{a0} \]  
\[ (5.24) \]

and

\[ L_{sp} = \begin{bmatrix} L_{sid} & L_{siq} & \ldots & \ldots \\ L_{sld} & L_{sld} & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ L_{sld} & L_{sld} & \ldots & L_{sld} \end{bmatrix} \]

In equation (5.23) \( \psi_d \) and \( \psi_q \) may be regarded as the flux linkages in fictitious windings moving with the rotor and centered over the direct and quadrature axes, respectively. \( L_d \) and \( L_q \) are self-inductances of these equivalent armature circuits.

In a similar manner, the armature voltage equations become

\[ T^{-1} e_{dq0} = \frac{d}{dt} (T^{-1} \Psi_{d0}) - r T^{-1} i_{dq0} \]

or

\[ e_{dq0} = T \frac{d}{dt} (T^{-1} \Psi_{d0}) - r i_{dq0} \]  
\[ (5.25) \]

After carrying out the indicated matrix operation, the above reduces to

\[ e_{dq0} = \frac{d}{dt} \Psi_{d0} + \frac{d\theta}{dt} S \Psi_{d0} - r i_{dq0} \]  
\[ (5.26) \]

where

\[ S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

We note that equation (5.26) is similar to the original equation (5.3), but with the addition of the generated or speed voltage terms, \( \frac{d\theta}{dt} S \Psi_{d0} \).

The complete set of machine performance equations now consists of the circuit voltage equations (5.26) and (5.5), and the flux-linkage equations (5.23) and (5.21). At constant rotor speed these equations are linear differential equations with constant coefficients, and even with variable rotor speed they are considerably simpler than the original set of equations.

The phase quantities may be found from the transformed variables by applying the inverse transformation as indicated in equation (5.17).

**Per Unit System**

In order for the equations describing the synchronous machine to be expressible in terms of an electrical equivalent circuit the mutual inductance coefficients must be reciprocal. Inspection of equations (5.21) and (5.23) shows that this condition is not completely satisfied in the present form of the equations. This difficulty arises because of the particular transformation used.
Several other transformations have been reported in the literature in an attempt to avoid this problem, and also to render the transformation power invariant, in which the expression for power is of the same form in both the original and transformed variables. The transformations differ only in the constants used. It may be noted that in order for the transformation to be power invariant the transformation matrix must be orthogonal (i.e., $T^{-1} = T'$). The original transformation used by Park has been retained here, since the same effects can be achieved by a suitable choice of per unit system, noting that for balanced operation $e_o = i_o = 0$.

In the basic equations currents and voltages have been expressed as instantaneous values. In the case of stator sinusoidal quantities, these have been expressed in terms of the peak values of the sine wave. In the choice of the base quantities, once voltage, current and frequency bases are picked, the bases for the remaining variables or circuit parameters such as flux-linkages, inductance, etc. are automatically set, such as:

$$
\Psi_{\text{base}} = \frac{e_{\text{base}}}{\omega_{\text{base}}},
$$

$$
L_{\text{base}} = \frac{\Psi_{\text{base}}}{i_{\text{base}}} = \frac{e_{\text{base}}}{\omega_{\text{base}} i_{\text{base}}},
$$

Also, in deriving the per unit quantities, the variables should be divided by their appropriate bases. For instance, if the value of voltage is a peak value, in order to express it in per unit it should be divided by the peak voltage base.

The flux linkage equations (5.21) and (5.23) can be expressed as

$$
\begin{bmatrix}
\Psi_d \\
\Psi_q
\end{bmatrix}
= \begin{bmatrix}
\Lambda_d & I_d \\
\Lambda_q & I_q
\end{bmatrix}
\begin{bmatrix}
I_d \\
I_q
\end{bmatrix}
$$

where

$$
\Psi_d = \begin{bmatrix}
\Psi_d \\
\Psi_{fd} \\
\Psi_{ld}
\end{bmatrix}, \quad \Psi_q = \begin{bmatrix}
\Psi_q \\
\Psi_{fq} \\
\Psi_{lq}
\end{bmatrix}, \quad I_d = \begin{bmatrix}
i_d \\
i_{fd} \\
i_{ld}
\end{bmatrix}, \quad I_q = \begin{bmatrix}
i_q \\
i_{fq} \\
i_{lq}
\end{bmatrix},
$$

$$
\Lambda_d = \begin{bmatrix}
-L_d & L_{afd} & L_{a1d} & \cdots \\
-L_{afd} & L_{ffd} & L_{f1d} & \cdots \\
-L_{a1d} & L_{1fd} & L_{11d} & \cdots
\end{bmatrix}, \quad \text{and} \quad \Lambda_q = \begin{bmatrix}
-L_q & L_{alq} & \cdots & \cdots \\
-\frac{3}{2} L_{alq} & L_{llq} & \cdots & \cdots \\
-\frac{3}{2} L_{llq} & \cdots & \cdots & \cdots
\end{bmatrix}.
$$
The base quantities are defined as follows:

- \( i_{ao} \) = peak rated phase current (amps)
- \( e_{ao} \) = peak rated phase voltage (volts)
- \( Z_{ao} = e_{ao}/i_{ao} \) (ohms)
- \( \omega_o, f_o \) = rated frequency (rad/sec, cyc/sec)
- \( L_{ao} = Z_{ao}/\omega_o \) (henries)
- \( X_{ao} = Z_{ao} = \omega_o L_{ao} \)

- \( \psi_{ao} \) = stator base flux linkage (weber turns)
- \( 3\varphi_{VA} \) base = \( VA_o = (3/2)e_{ao}i_{ao} = (3/2)\omega_o L_{ao}i_{ao} = (3/2)\omega_o \psi_{ao}i_{ao} \)
- \( \psi_{fdo} \) = base field flux linkage
- \( \psi_{ado} \) = base flux linkage for additional \( d \) axis rotor windings
- \( \psi_{xqo} \) = base flux linkage for additional \( q \) axis rotor windings

\( I_{fdo}, I_{ado}, I_{xqo} \) are the base currents of the corresponding rotor windings.

It is not possible to assign physical values to the rotor base quantities at this stage.

In order to express equation (5.27) in per unit form, a diagonal matrix \( \Psi_o \) of base flux linkages and another \( I_o \) of base currents are defined and these are operated on (5.27) to give

\[
\begin{bmatrix}
\Psi_{d pu} \\
\Psi_{q pu}
\end{bmatrix} = \Psi_o^{-1} \begin{bmatrix}
\Lambda_d \\
\Lambda_q
\end{bmatrix} \begin{bmatrix}
I_{d pu} \\
I_{q pu}
\end{bmatrix}
\]

which, after removing the subscript pu for convenience, can be written in short as

\[
\Psi = \Lambda I \quad (5.28)
\]

For the per unit system of equations to have reciprocal mutual inductances, the matrix \( \Lambda \) in the above equation must be made symmetrical. Equating the corresponding off diagonal elements and rearranging, the following relations are obtained.

- \( \alpha_o \; \psi_{fdo} I_{fdo} = (3/2)\alpha_o \; \psi_{ado} i_{ao} = VA_o \quad (5.29) \)
- \( \alpha_o \; \psi_{ado} I_{ado} = (3/2)\alpha_o \; \psi_{xqo} i_{xqo} = VA_o \quad (5.30) \)
- \( \alpha_o \; \psi_{xqo} I_{xqo} = (3/2)\alpha_o \; \psi_{fdo} i_{fdo} = VA_o \quad (5.31) \)

This shows that in order to have reciprocal mutual inductances the volt-ampere bases of all the rotor circuits must be the same as the machine volt-ampere base.

Since with the rated frequency used as the base frequency, per unit inductance becomes the same as per unit reactance, it is customary to use the reactance symbol \( x \) to denote inductances in the per unit machine equations.

Also, since \( \theta = \alpha_o t \pm \theta_o, \frac{d\theta}{dt} = \omega \)

Thus the per unit equation (5.28) can be written in expanded form as
\[
\begin{align*}
\begin{bmatrix}
\psi_d \\
\psi_{fd} \\
\psi_{1d} \\
\vdots \\
\psi_q \\
\psi_{1q} \\
\vdots \\
\end{bmatrix}
& =
\begin{bmatrix}
-X_d & X_{afd} & X_{a1d} & \cdots \\
-X_{fad} & X_{fdd} & X_{f1d} & \cdots \\
-X_{1ad} & X_{1fd} & X_{11d} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
-X_q & X_{a1q} & \cdots & i_q \\
-X_{1aq} & X_{11q} & \cdots & i_{1q} \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_{fd} \\
i_{1d} \\
\vdots \\
i_q \\
i_{1q} \\
\vdots \\
\end{bmatrix}
\end{align*}
\]

where all the quantities are in per unit. \(x_d, x_{afd}, \text{etc.}\) can be shown, using equations (5.29) – (5.31), to be given by

\[
x_{afd} = x_{fad} = \frac{3}{2} \frac{L_{afd}}{L_{ao}} \left( \frac{I_{fdo}}{\frac{3}{2} i_{ao}} \right)^2
\]

\[
x_{a1d} = x_{1ad} = \frac{3}{2} \frac{L_{a1d}}{L_{ao}} \left( \frac{I_{xdo}}{\frac{3}{2} i_{ao}} \right)^2
\]

e tc.

\[
x_{fdd} = \frac{3}{2} \frac{L_{fdd}}{L_{ao}} \left( \frac{I_{fdo}}{\frac{3}{2} i_{ao}} \right)^2
\]

\[
x_{f1d} = x_{1fd} = \frac{3}{2} \frac{L_{f1d}}{L_{ao}} \left( \frac{I_{xdo} I_{fdo}}{\frac{9}{4} i_{ao}^2} \right)^2
\]

e tc.

\[
x_{11d} = \frac{3}{2} \frac{L_{11d}}{L_{ao}} \left( \frac{I_{xdo}}{\frac{3}{2} i_{ao}} \right)^2
\]

\[
x_{12d} = x_{21d} = \frac{3}{2} \frac{L_{12d}}{L_{ao}} \left( \frac{I_{xdo}}{\frac{3}{2} i_{ao}} \right)^2
\]

e tc.
These are the conversion formulae for converting the impedances from normal units to per unit values. The form in which the expressions have been written is to be particularly noted. Each expression has been written in terms of base current ratios. The base current ratios have been taken as the ratio of the base rotor circuit currents to 3/2 the base stator current. Also each of the original impedance values has been multiplied by 3/2. This is necessary to give uniform conversion formulas for all the self and mutual impedances while converting the original non-reciprocal system of equations to a reciprocal per unit system of equations. The selection of base current ratios is equivalent to the selection of stator rotor turns ratio. In other words, all the quantities are first referred to the stator using the specially formed turns ration and multiplying by 3/2 and then the stator base quantities are used to find the per unit values. As there is no unique choice of selecting the turns ratio there is no unique value for the per unit impedance that has been derived by applying the turns ratio. The base rotor currents are usually selected so as to make all per unit values of self and mutual impedances of the same order of magnitude. One choice of selecting base rotor currents is discussed later.

Dividing through the individual voltage equations in (5.5) and (5.26) by the respective base quantities and manipulating a little, the per unit form of the equations can be expressed in matrix form as

\[
V_{r \text{ pu}} = \frac{1}{\omega_o} \frac{d}{dt} \Psi_{r \text{ pu}} + R_{r \text{ pu}} I_{r \text{ pu}}
\]

\[
e_{d\text{pu}} = \frac{1}{\omega_o} \frac{d}{dt} \Psi_{d\text{pu}} + \frac{\omega}{\omega_o} S \Psi_{d\text{pu}} - r_{\text{pu}} i_{d\text{pu}}
\]

where the per unit quantities are given by

\[
e_{f\text{du}} = \frac{e_{fd}}{e_o} \frac{I_{f\text{du}}}{3 i_{ao}}
\]
SYNCHRONOUS MACHINES

\[ \psi_{fd} = \psi_{ao} \left( \frac{I_{fdo}}{\frac{3}{2} i_{ao}} \right) \]  \hspace{1cm} (5.46)

\[ r_{fd} = \frac{3 r_{fd}}{2 r_{ao}} \left( \frac{I_{fdo}}{\frac{3}{2} i_{ao}} \right)^2 \]  \hspace{1cm} (5.47)

\[ r_{1d} = \frac{3 r_{1d}}{2 r_{ao}} \left( \frac{I_{xdo}}{\frac{3}{2} i_{ao}} \right)^2 \]  \hspace{1cm} (5.48)

etc.

\[ r_{pu} = \frac{r}{r_{ao}} \]  \hspace{1cm} (5.49)

\[ e_{d} = \frac{e_{d}}{e_{ao}} \]  \hspace{1cm} (5.50)

etc.

\( x_d'', x_q'', x_d' \) are pu impedances (defined later) as seen from the machine terminals and so they must be independent of the base rotor currents and will be dependent only on the machine volt ampere and voltage base.

\( x_d'', x_q'', x_d' \), as derived later, are given by, considering one additional rotor circuit on each axis,

\[ x_d'' = x_d - \frac{x_{11d} x_{afd}^2 - 2 x_{f1d} x_{a1d} x_{afd} + x_{f1d} x_{afd}^2}{x_{11d} x_{f1d} - x_{f1d}^2} \]  \hspace{1cm} (5.51)

\[ x_q'' = x_q - \frac{x_{a1q}^2}{x_{11q}} \]  \hspace{1cm} (5.52)

\[ x_d' = x_d - \frac{x_{afd}^2}{x_{f1d}} \]  \hspace{1cm} (5.53)

Substituting the values of \( x_d, x_{11d} \) etc. as found previously,

\[ x_d'' = \frac{L_d}{L_{ao}} - \frac{3}{2} \frac{L_{11d} L_{afd}^2 - 2 L_{f1d} L_{a1d} L_{afd} + L_{f1d} L_{afd}^2}{L_{11d} L_{f1d} - L_{f1d}^2} \]  \hspace{1cm} (5.54)

\[ x_q'' = \frac{L_q}{L_{ao}} - \frac{3}{2} \frac{L_{a1q}^2}{L_{11q}} \]  \hspace{1cm} (5.55)
\[ x_d' = \frac{L_d}{L_{ao}} - \frac{3}{2} \frac{1}{L_{ao}} \frac{L_{af}^2}{L_{f}^{ad}} \]  

(5.56)

\( x_d'' \), \( x_q'' \), \( x_d' \) are thus invariant when expressed in per unit of the machine volt ampere and voltage base.

**Choice of rotor base currents**

Several definitions of rotor base currents are available. The base field current which is most commonly used is that current which will induce in each stator phase a voltage equal to \( \omega_o L_{ad} i_{ao} \) or \( X_{ad} i_{ao} \). \( L_{ad} \) is the mutual component of the stator self-inductance \( L_{ds} \), i.e., the component of \( L_d \) corresponding to the flux linkages which link the rotor. Defining a leakage component \( L_{1d} \) corresponding to the flux linkages which do not link the rotor,

\[ L_d = L_{ad} + L_{1d} \]

Similarly,

\[ L_q = L_{aq} + L_{1q} \]

The leakage components in the \( d \) and \( q \) axes are generally assumed equal and labeled \( L_l \). Let base currents of other rotor circuits be similarly defined. With these definitions

\[
\begin{align*}
L_{af} I_{fdo} &= L_{ad} i_{ao} \\
L_{al} I_{alo} &= L_{ad} i_{ao} \\
L_{alq} I_{alo} &= L_{ad} i_{ao}, \text{ etc.}
\end{align*}
\]

Multiplying both sides of the first equation by \( \frac{3}{2} \frac{I_{fdo}}{\frac{3}{2} i_{ao} L_{ao}} \), we have

\[ \frac{3}{2} \frac{I_{fdo}}{\frac{3}{2} i_{ao} L_{ao}} L_{af} I_{fdo} = \frac{3}{2} \frac{I_{fdo}}{\frac{3}{2} i_{ao} L_{ao}} L_{ad} i_{ao} \]

or

\[ x_{af} I_{fdo} = x_{ad} I_{fdo} \]

or

\[ x_{af} = x_{ad} \]

Similarly, from the second and third equation

\[ x_{al} = x_{ad}, x_{alq} = x_{aq} \]

Thus this definition makes \( x_{af} \) and \( x_{al} \) equal to \( x_{ad} \). It cannot, however, be said that \( x_{f} \) is also equal to \( x_{ad} \), although due to the proximity of the rotor circuits to the air gap this is very nearly true, and in machine analysis it is often assumed that in per unit form all the direct axis mutual inductances are equal and all the quadrature axis mutual inductances are equal. Therefore we can write
SYNCHRONOUS MACHINES

\[ x_{ad} = x_{a0d} = x_{a1d} = x_{a2d} = \cdots = x_{f1d} = \cdots \]
\[ x_{aq} = x_{a0q} = x_{a2q} \cdots \]
\[ x_{fd} = x_{ad} + x_{fd} \]
\[ x_{11d} = x_{ad} + x_{1d} \]
\[ \vdots \]
\[ x_{11q} = x_{aq} + x_{1q} \]
\[ \vdots \]
\[ x_{d} = x_{ad} + x_{l} \]
\[ x_{q} = x_{aq} + x_{l} \]

(5.57)

To find the base field current, let \( I_{f0} \) be the field current which generates rated stator voltage on open circuit at rated speed as found from the air gap line. Then

\[ I_{f0} \times \omega L_{a0d} = \omega L_{a0} i_{a0} \]

Also

\[ I_{f0} \times L_{a0d} = L_{a0} i_{a0} \]

From these

\[ I_{f0} / I_{f0} = L_{a0d} / L_{a0} = x_{ad} \]

or

\[ I_{f0} = x_{ad} \times I_{f0} \]

(5.58)

The per unit equations can be summarized as follows:

**Per unit voltage equations**

**Stator**

\[
\begin{bmatrix}
  e_d \\
  e_q \\
  e_o
\end{bmatrix} = \frac{1}{\omega_o} \frac{d}{dt} \begin{bmatrix}
  \psi_d \\
  \psi_q \\
  0
\end{bmatrix} + \frac{\omega}{\omega_o} \begin{bmatrix}
  -\psi_q \\
  \psi_d \\
  0
\end{bmatrix} - \begin{bmatrix}
  r_d \\
  r_q \\
  r_o
\end{bmatrix} \begin{bmatrix}
  i_d \\
  i_q \\
  i_o
\end{bmatrix}
\]

(5.59)

**Rotor**

\[
\begin{bmatrix}
  e_{fd} \\
  0 \\
  \vdots \\
  0
\end{bmatrix} = \frac{1}{\omega_o} \frac{d}{dt} \begin{bmatrix}
  \psi_{fd} \\
  \psi_{1d} \\
  \vdots \\
  \psi_{1q}
\end{bmatrix} + \begin{bmatrix}
  r_{fd} \\
  r_{1d} \\
  \vdots \\
  r_{1q}
\end{bmatrix} \begin{bmatrix}
  i_{fd} \\
  i_{1d} \\
  \vdots \\
  i_{1q}
\end{bmatrix}
\]

(5.60)
Per unit flux-linkage equations

**Stator**

\[
\begin{bmatrix}
\psi_d \\
\psi_q \\
\psi_o
\end{bmatrix} = \begin{bmatrix} x_d & x_{d1} & \cdots \\ x_q & x_{q1} & \cdots \\ x_o & x_{o1} & \cdots 
\end{bmatrix} \begin{bmatrix} i_d \\
\end{bmatrix} + \begin{bmatrix} x_{afd} & x_{a1d} & \cdots \\
\end{bmatrix} \begin{bmatrix} i_f \end{bmatrix} (5.61)
\]

**Rotor**

\[
\begin{bmatrix}
\psi_{fd} \\
\psi_{1d} \\
\psi_{1q} \\
\psi_{2q} \\
\vdots
\end{bmatrix} = \begin{bmatrix} x_{afd} & x_{a1d} & \cdots \\ x_{fd1} & x_{f1d} & \cdots \\ \vdots & \vdots & \cdots \\
\end{bmatrix} \begin{bmatrix} i_d \\
\end{bmatrix} + \begin{bmatrix} x_{faq} & x_{f1q} & \cdots \\
\end{bmatrix} \begin{bmatrix} i_f \\
\end{bmatrix} (5.62)
\]

In the above equations all quantities are in per unit except time which is in seconds, and \( \omega \) and \( \omega_o \), which are in rad/sec.

**Power and Torque**

The instantaneous power measured at the machine terminals is given by

\[
P = e'_{abc} i_{abc} (5.63)
\]

In terms of \( dqo \) components

\[
P = e'_{dqo} [T^{-1}]T^{-1} i_{dqo} = \frac{3}{2} e'_{dqo} \begin{bmatrix} 1 \\
1
\end{bmatrix} i_{dqo} (5.64)
\]

Under normal balanced condition \( e_o = i_o = 0 \)

\[
\therefore P = \frac{3}{2} (e_d i_d + e_q i_q) (5.65)
\]

Substituting for \( e_d \) and \( e_q \) from equation (5.26)

\[
P = \frac{3}{2} \left[ (\psi_d - \omega \psi_q - r_i d i_d) + (\psi_q + \omega \psi_d - r_i q i_q) \right]
\]

\[
= \frac{3}{2} \left[ (\psi_d i_d + \psi_q i_q) + \omega (\psi_q i_q - \psi_d i_d) - r(i_d^2 + i_q^2) \right] (5.66)
\]

The above equation can be interpreted as: Net power output = (rate of decrease of armature magnetic energy) + (power transferred across air-gap) – (armature resistance loss). The air-gap torque is obtained by dividing the second term (air-gap power) by rotor speed. Thus

\[
T_e = \frac{3}{2} (\psi_d i_q - \psi_q i_d) (5.67)
\]
For balanced operation with zero armature resistance, \( e_d = -\omega \psi_q \) and \( e_q = \omega \psi_d \), and the torque is

\[
T_e = \frac{3}{2} \frac{1}{\omega} (e_q i_d + e_q i_q)
\]

which checks with equation (5.65)

Expressed in per unit of three phase power base, equation (5.65) becomes

\[
P = e_d i_d + e_q i_q \tag{5.68}
\]

Similarly, the expression for per unit torque is

\[
T_e = \psi_d i_q - \psi_q i_d \tag{5.69}
\]

**Steady State Operation**

Consider a synchronous machine operating as a generator, supplying power to an infinite bus. Let the machine phase voltages be

\[
e_a = e \cos \omega t
\]

\[
e_b = e \cos (\omega t - 120°)
\]

\[
e_c = e \cos (\omega t + 120°)
\]

Let the phase currents supplied by the generator be

\[
i_a = i \cos (\omega t - \phi)
\]

\[
i_b = i \cos (\omega t - 120° - \phi)
\]

\[
i_c = i \cos (\omega t + 120° - \phi)
\]

We apply the \(dqo\) transformation to the voltages using equation (5.20), noting that in matrix \( T \), \( \theta \) is the angle between the direct axis and the axis of phase \( a \). In the steady state the rotor electrical speed \( \omega \) is the same as the frequency of the voltage and current waves. If we assume that at \( t = 0 \) the \( d \) axis lags phase \( a \) by \( \theta_o \), then \( \theta = \omega t - \theta_o \). Substituting the expressions for the phase voltages in (5.20),

\[
e_d = e \cos \theta_o
\]

\[
e_q = e \sin \theta_o \tag{5.72}
\]

Similarly, for the current,

\[
i_d = i \cos(\theta_o - \phi)
\]

\[
i_q = i \sin(\theta_o - \phi) \tag{5.73}
\]

Equations (5.72) and (5.73) show that for balanced steady state operation, \( e_d, e_q, i_d \) and \( i_q \) are constant dc quantities.

Under steady state conditions all \( d \) and \( q \) axes flux-linkages are constants, i.e., their rates of changes are zero. Therefore, since under steady state \( \omega = \omega_o \), we obtain from equations (5.59) to (5.62)
SYNCHRONOUS MACHINES

\[ e_d = -\psi_q - r i_d \]  (5.74)
\[ e_q = \psi_d - r i_q \]
\[ e_{fd} = r_{fd} i_{fd} \]
\[ i_{id} = i_{iq} = \ldots = 0 \]  (5.75)
\[ \psi_d = -x_d i_d + x_{ad} i_{fd} = -x_d i_d + x_{ad} i_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd} \]
\[ \psi_q = -x_q i_q \]
\[ \psi_{fd} = -x_{afd} i_d + x_{qfd} i_{fd} \]  (5.76)

From (5.74) and (5.76) we obtain
\[ e_d = x_q i_q - r i_d \]
\[ e_q = E - x_d i_d - r i_q \]  (5.77)

where
\[ E = x_{ad} i_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd} \]

\( E \) is introduced for convenience and may be considered as the field excitation measured in terms of the terminal voltage that it would produce on open-circuit, normal-speed operation.

Observation of equations (5.77) shows that it is possible to consider the voltages and currents as phasors in a plane having \( d \) and \( q \) coordinate axes mutually perpendicular and oriented in exactly the same way as the \( d \) and \( q \) axes of the machine itself as shown in Figure 5.1. (Note that these axes rotate in space at the electrical speed of the rotor, and in this sense they are conceptually related to the phasor representing an ac quantity which rotates at synchronous speed.)

Complex voltage and current may be defined as
\[ \hat{e} = e_d + j e_q \]
\[ \hat{i} = i_d + j i_q \]  (1)  (5.78)

and from (5.72) and (5.73)
\[ \hat{e} = e_d + j e_q = e \cos \theta_o + j e \sin \theta_o \]
\[ \hat{i} = i_d + j i_q = i \cos(\theta_o - \phi) + j i \sin(\theta_o - \phi) \]  (5.79)

Note that these equations were derived from a definition of \( e_a = e \cos \omega t \) and \( i_a = i \cos(\omega t - \phi) \), and \( \theta = \omega t - \theta_o \), the angle between the direct axis and the axis of phase \( a \). At \( t = 0 \), the \( d \) axis lags phase \( a \) by \( \theta_o \). Hence, expressing \( e_a \) on the \( d-q \) axes plane is equivalent to lining up the \( e_a \) phasor with \( e \) as given by equation (5.79), from which it follows that \( e_a \) leads the \( d \) axis by \( \theta_o \). Note also that the phase displacement between \( e \) and \( i \) is the same as that between \( e_a \) and \( i_a \).

(1) a “\(^\wedge\)” over a symbol is used to distinguish a phasor from a scalar quantity. This will, however, be omitted when there is no chance of confusion.
Locating the $d$ and $q$ axes

Rewriting equations (5.77) as

$$e_d = x_q i_q - r i_d$$

$$e_q = E - (x_d - x_q) i_d - x_q i_d - r i_q = E - x_q i_d - r i_q$$

(5.80)

where

$$E_q = E - (x_d - x_q) i_d$$

(5.81)

and then combining them in terms of the complex quantities defined in (5.78)

$$e_d + j e_q = j E_q - (r + j x_q)(i_d + j i_q)$$

(5.82)

Equation (5.82) shows that by adding the $(r + j x_q)(i_d + j i_q)$ voltage drop to the terminal voltage $(\hat{e} = e_d + j e_q)$ we obtain a quantity $E_q$ which lies along the quadrature axis and which is related to the excitation $E$ by equation (5.81). With the quadrature axis located, $i$ can be resolved into its $d$ and $q$ components, and $(x_d - x_q) i_d$ added to $E_q$ to find $E$. The complete procedure is illustrated in Figure 5.2

![Fig. 5.2 Steady-state phasor diagram of a synchronous generator.](image)

Note that the relationship $E = x_{ad} i_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd}$ (see equations 5.76 and 5.77) is true only in the steady state.

**Open-circuit operation**

Under open-circuit condition

$$i_d = i_q = 0$$

$$\therefore e_q = E_q = E = x_{ad} i_{fd} = \hat{e}$$

(5.83)

and

$$e_d = 0$$

It may be noted that the quantity $x_{ad} i_{fd} (= E)$ in per unit has a magnitude comparable to that of voltage. Hence, rather than dealing with per unit field current, $i_{fd}$, it is convenient and customary to deal with $x_{ad} i_{fd}$ in per unit.
In the per unit system adopted here, in order to have reciprocal mutuals, the volt-ampere base of the field had to be equal to the three phase stator volt-ampere base. Since in actual operation the field volt-ampere is in the order of 0.5% of the stator volt-ampere, the value of per unit $e_{fd}$ for typical operating condition would be in the order of 0.005pu. A new per unit field voltage is therefore defined as

$$E_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd}$$

(5.84)

This new variable is proportional to the original per unit field voltage $e_{fd}$. We now have the following steady state relationship

$$E_{fd} = E = x_{ad} i_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd}$$

(5.85)

With the variable $E$ defined as $E = x_{ad} i_{fd}$ (see equation 5.77), it can be seen that when $i_{fd} = 1/x_{ad}$, $E = 1.0$ pu. On open-circuit, neglecting saturation, when $E = 1.0$, $e_d$ (per unit terminal voltage) = 1.0. $E$ is proportional to the field current, not to the field voltage, except in the steady state.

**Reactive Power**

$$P = e_d i_d + e_q i_q$$

From Figure 5.2, we can write

$$P = e \sin \delta i \sin(\delta + \phi) + e \cos \delta i \cos(\delta + \phi) = e i \cos \phi$$

Since the phase displacement between $\hat{e}$ and $\hat{i}$ is the same as that between the phase quantities, we can write, from the definition of reactive power,

$$Q = e i \sin \phi = e i \sin(\delta + \phi - \delta) = e i[\sin(\delta + \phi) \cos \delta - \cos(\delta + \phi) \sin \delta]$$

$$Q = e_d i_d - e_q i_q$$

\( \therefore P + jQ = (e_d i_d + e_q i_q) + j(e_q i_d - e_d i_q) = (e_d + j e_q )(i_d - j i_q) = \hat{e}\hat{i}^* \)

(5.87)

**Subtransient and Transient Reactances and Time Constants**

In this discussion only one damper winding on each axis is assumed. With all rotor circuits short circuited, let voltages be suddenly applied to the stator so that only $d$ axis current flows. Immediately after the voltage applied, the flux linkages $\psi_{fd}$ and $\psi_{1d}$ cannot change and so are still zero. Therefore, from (5.61) and (5.62), at $t = 0^+$,

$$\psi_d = -x_d i_d + x_{qf} i_{fd} + x_{af} i_{1d}$$

(5.88)

$$\psi_{fd} = 0 = -x_{qf} i_d + x_{d} i_{fd} + x_{f1} i_{1d}$$

(5.89)

$$\psi_{1d} = 0 = -x_{af} i_d + x_{f1} i_{fd} + x_{11} i_{1d}$$

(5.90)

Obtaining $i_{fd}$ and $i_{11d}$ in terms of $i_d$ from (5.89) and (5.90), and substituting in (5.88),
\[
\psi_d = -\left( x_d - \frac{x_{11d}^2 x_{ad}^2 - 2 x_{11d} x_{ad} x_{afld} + x_{fild} x_{ad}^2}{x_{11d} x_{fild} - x_{f1d}^2} \right) i_d
\]

Since subtransient inductance (reactance) is defined as the initial flux linkage per unit of stator current, the \( d \) axis subtransient reactance is given by

\[
x_d'' = x_d - \frac{x_{11d}^2 x_{ad}^2 - 2 x_{11d} x_{ad} x_{afld} + x_{fild} x_{ad}^2}{x_{11d} x_{fild} - x_{f1d}^2}
\]

(5.91)

Using the relationships (5.57), the above can also be written as

\[
x_d'' = x_i + \frac{x_{ad} x_{fd} x_{1d}}{x_{ad} (x_{fd} + x_{1d}) + x_{fild} x_{1d}}
\]

(5.92)

Damper winding currents decay much faster than the field winding current. The \( d \) axis transient reactance is obtained by assuming that the damper winding current has decayed to zero while the field flux linkage is still zero. Thus

\[
\psi_d = -x_d i_d + x_{ad} i_{fd}
\]

(5.93)

\[
\psi_{fd} = 0 = -x_{ad} i_d + x_{fild} i_{fd}
\]

(5.94)

Obtaining \( i_{fd} \) from (5.94) and substituting in (5.93)

\[
\psi_d = -\left( x_d - \frac{x_{ad}^2}{x_{fild}} \right) i_d
\]

Thus, the transient reactance is

\[
x_d' = x_d - \frac{x_{ad}^2}{x_{fild}}
\]

(5.95)

which can also be written as

\[
x_d' = x_i + \frac{x_{ad} x_{fd}}{x_{ad} + x_{fild}}
\]

(5.96)

The \( q \) axis subtransient reactance can be obtained in a similar manner. At \( t = 0^+ \)

\[
\psi_q = -x_q i_q + x_{a1q} i_{1q}
\]

(5.97)

\[
\psi_{1q} = 0 = -x_{a1q} i_q + x_{11q} i_{1q}
\]

(5.98)

Obtaining \( i_{1q} \) from (5.98) and substituting in (5.97)

\[
\psi_q = -\left( x_q - \frac{x_{a1q}}{x_{11q}} \right) i_q
\]

Thus

\[
x_q'' = x_q - \frac{x_{a1q}^2}{x_{11q}} = x_i + \frac{x_{aq} x_{1q}}{x_{aq} + x_{1q}}
\]

(5.99)
Since we have assumed one damper winding on the \( q \) axis and since there is no \( q \) axis field winding, the \( q \) axis transient reactance is the same as the \( q \) axis synchronous reactance. In round rotor machine, due to the multiple paths of circulating eddy currents, it may be necessary to consider more than one damper winding on the \( q \) axis. Then there will be distinct \( q \) axis subtransient, transient, and synchronous reactances.

**Time constants**

To obtain the \( d \) axis open circuit time constants, consider a voltage applied to the field with the stator open circuited. Since \( i_d = 0 \), we can write the rotor flux linkage and voltage equations as

\[
\psi_{fd} = x_{fd}i_{fd} + x_{f1d}i_{id} \tag{5.100}
\]

\[
\psi_{id} = x_{f1d}i_{fd} + x_{11d}i_{id} \tag{5.101}
\]

\[
e_{fd} = \frac{1}{\omega_o} \frac{d}{dt} \psi_{fd} + r_{fd}i_{fd} \tag{5.102}
\]

\[
0 = \frac{1}{\omega_o} \frac{d}{dt} \psi_{id} + r_{id}i_{id} \tag{5.103}
\]

At \( t = 0^+ \), \( \psi_{id} = 0 \), and therefore

\[
i_{fd} = -\frac{x_{11d}}{x_{f1d}} i_{id} \tag{5.104}
\]

Substituting (5.101) in (5.103), we obtain

\[
\frac{x_{f1d}}{\omega_o} \frac{d}{dt} i_{fd} = -\frac{x_{11d}}{\omega_o} \frac{d}{dt} i_{id} - r_{id}i_{id} \tag{5.105}
\]

Substituting (5.100) in (5.102), and using (5.104) and (5.105),

\[
e_{fd} = -\frac{x_{11d}x_{fd} - x_{f1d}^2}{\omega_o x_{f1d}} \frac{d}{dt} i_{id} = \frac{x_{f1d}r_{fd} + x_{11d}r_{fd}}{x_{f1d}} i_{id} \tag{5.106}
\]

Therefore, the \( d \) axis open-circuit subtransient time constant is

\[
T_{do} = \frac{x_{11d}x_{fd} - x_{f1d}^2}{\omega_o (x_{fd}r_{fd} + x_{11d}r_{fd})} \tag{5.107}
\]

Since \( r_{fd} \ll r_{id} \), the first term within the parenthesis in the denominator can be neglected, so that

\[
T_{do} = \frac{x_{11d}x_{fd} - x_{f1d}^2}{\omega_o x_{fd}r_{1id}} = \frac{x_{11d} - x_{f1d}^2}{\omega_o r_{fd}^2} \tag{5.107}
\]
Note that the same expression for $T_{do}'$ can also be obtained from the damper circuit voltage equation, with the field short circuited. Then, at $t = 0^+$, $\Delta \psi_{fd} = 0$, and $\Delta i_{fd} = -\frac{x_{fd}}{x_{fd}} \Delta i_{id}$.

Substituting in (5.105)

$$0 = \frac{1}{\omega_o} \left( x_{11d} - x_{f1d}^2 x_{fd} \right) \frac{d}{dt} \Delta i_{id} + r_{id} \Delta i_{id}$$

from which

$$T_{do}'' = \frac{x_{11d} - x_{f1d}^2 x_{fd}}{\omega_o r_{id}}$$

The $d$ axis open-circuit transient time constant is obtained by either assuming that the damper circuit current has decayed to zero or the damper winding is absent. Then we have

$$\psi_{fd} = x_{fd} i_{fd}$$

and

$$e_{fd} = x_{fd} \frac{d}{dt} i_{fd} + r_{fd} i_{fd}$$

Therefore, the $d$ axis open-circuit transient time constant is

$$T_{do}' = \frac{x_{fd}}{\omega_o r_{fd}}$$

(5.108)

The short-circuit time constants can be obtained by assuming that the stator circuits are short circuited. Thus, to obtain the $d$ axis short-circuit subtransient time constant,

$$e_q = 0 = -x_d i_d + x_{qfd} i_{fd} + x_{qmd} i_{id}$$

(5.109)

(neglecting armature resistance and the $\psi_q$ term)

$$\psi_{fd} = -x_{qfd} i_d + x_{fd} i_{fd} + x_{f1d} i_{id}$$

(5.110)

$$\psi_{id} = -x_{a1d} i_d + x_{f1d} i_{fd} + x_{11d} i_{id}$$

(5.111)

Substituting $i_d$ from (5.109) into (5.110) and (5.111)

$$\psi_{fd} = \left( x_{fd} - \frac{x_{qfd}^2}{x_d} \right) i_{fd} + \left( x_{f1d} - \frac{x_{a1d} x_{qfd}}{x_d} \right) i_{id}$$

(5.112)

$$\psi_{id} = \left( x_{f1d} - \frac{x_{a1d} x_{qfd}}{x_d} \right) i_{fd} + \left( x_{11d} - \frac{x_{a1d}^2}{x_d} \right) i_{id}$$

(5.113)

Comparing (5.112) and (5.113) with (5.100) and (5.101), we can write the $d$ axis short-circuit subtransient time constant as
\[ T'^d = \frac{X_{f/d} - X_{a/fd}^2}{X_d} \left( X_{1/d} - X_{a/id}^2 \right) - \left( X_{f/1d} - X_{a/id} X_{a/fd} \right)^2 \]

\[ \omega_o \omega'_{1/d} \left( X_{f/d} - X_{a/fd}^2 \right) \]

\[ = x_d \left( x_{1/d} x_{f/d} - x_{f/1d}^2 \right) - \left( x_{1/d} x_{a/fd}^2 - 2 x_{f/1d} x_{a/id} x_{a/fd} + x_{f/d} x_{a/fd}^2 \right) \]

\[ \omega_o \omega'_{1/d} \left( x_d x_{f/d} - x_{a/fd}^2 \right) \]

\[ = T'^d \frac{x_{d}^*}{x_d} \quad \text{(5.114)} \]

The \( d \) axis short-circuit transient time constant is obtained by assuming that the damper circuit is absent. Then, following the previous steps, the short-circuit transient time constant is

\[ T'^d = \frac{x_{f/d} - X_{a/fd}^2}{x_d} \left( x_{1/d} - X_{a/id}^2 \right) - \left( x_{f/1d} - X_{a/id} X_{a/fd} \right)^2 \]

\[ \omega_o \omega'_{1/d} \left( x_{f/d} - X_{a/fd}^2 \right) \]

\[ = \left( x_{1/d} x_{f/d} - x_{f/1d}^2 \right) \frac{x_{d}^*}{x_d} \]

\[ \omega_o \omega'_{1/d} \left( x_d x_{f/d} - X_{a/fd}^2 \right) \]

\[ = T'^d \frac{x_{d}^*}{x_d} \quad \text{(5.115)} \]

The \( q \) axis open-circuit and short-circuit time constants are obtained in a similar manner.

\[ T'^q = \frac{x_{1/q} x_{f/q}}{\omega_o \omega'_{1/q}} \quad \text{(5.116)} \]

\[ T'^q = T'^q \frac{x_{d}^*}{x_q} \quad \text{(5.117)} \]

**Synchronous Machine Models for Stability Studies**

The synchronous machine models suitable for stability studies will now be derived. We start with the voltage and flux linkage equations given by equations (5.59) - (5.62). Although these equations along with the equation for electrical torque (5.69) can be used directly, they would require unnecessarily short time step in the numerical integration without corresponding increase in accuracy. Therefore, except in specialized studies, the equations are simplified by neglecting the \( \psi_d \) and \( \psi_q \) terms, since their effect is usually small. Neglecting these terms is equivalent to neglecting the short lived power frequency transients that are observed in the machine variables immediately following a disturbance. This is also consistent with neglecting the transient terms.

5-24
Synchronous Machines

5-25

in transmission line modeling. Their impact on machine stability is not generally appreciable. In

general, they tend to provide some positive damping.

Rewriting the stator and rotor voltage equations (5.59) and (5.60), neglecting the \( \psi_d \) and \( \psi_q \)
terms, and noting that for balanced operation \( e_o = i_o = 0 \),

\[
V_s = W \psi_s + R_s I_s
\]

where

\[
V_s = \begin{bmatrix} e_d \\ e_q \end{bmatrix}, \quad \psi_s = \begin{bmatrix} \psi_d \\ \psi_q \end{bmatrix}, \quad I_s = \begin{bmatrix} i_d \\ i_q \end{bmatrix}
\]

\[
W = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad R_s = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}
\]

and

\[
\frac{1}{\omega_o} \frac{d}{dt} \psi_r = V_r + R_r I_r
\]

where

\[
\psi_r = \begin{bmatrix} \psi_{rd} \\ \psi_{1d} \\ \vdots \\ \psi_{1q} \end{bmatrix}, \quad V_r = \begin{bmatrix} e_{rd} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad I_r = \begin{bmatrix} i_{rd} \\ i_{1d} \\ \vdots \\ i_{1q} \end{bmatrix}, \quad R_r = \begin{bmatrix} -r_{rd} & 0 & \cdots & 0 \\ 0 & -r_{1d} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -r_{1q} \end{bmatrix}
\]

The flux-linkage equations, (5.61) and (5.62), can be rewritten as

\[
\psi_s = X_{s} I_s + X_{sr} I_r
\]

\[
\psi_r = X_{r} I_s + X_{rr} I_r
\]

where

\[
X_{ss} = \begin{bmatrix} -x_d & 0 \\ -x_q & 0 \end{bmatrix}, \quad X_{sr} = \begin{bmatrix} x_{rd} & x_{1d} & \cdots & \cdots \\ x_{rd} & x_{1d} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{rd} & x_{1d} & \cdots & x_{1q} \end{bmatrix}
\]

\[
X_{rs} = -X_{sr}^T, \quad X_{rr} = \begin{bmatrix} x_{rd} & x_{1d} & \cdots & \cdots \\ x_{rd} & x_{1d} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{rd} & x_{1d} & \cdots & x_{1q} \end{bmatrix}
\]

\( I_r \) obtained from equation (5.121) and substituted in (5.120) yields
From (5.122) and (5.118),

\[ V_s = \frac{1}{\omega_o} \frac{d}{dt} \Psi_s = V_r + \mathbf{R}_s \mathbf{X}_{rr}^{-1} \Psi_r - \mathbf{R}_r \mathbf{X}_{rr}^{-1} \mathbf{X}_{rs} \mathbf{I}_s \]  \hspace{1cm} (5.123)

Substituting the expression for \( I_r \) obtained from equation (5.121) into (5.119)

\[ \frac{1}{\omega_o} \frac{d}{dt} \Psi_r = V_r + \mathbf{R}_r \mathbf{X}_{rr}^{-1} \Psi_r \mathbf{X}_{rs} \mathbf{I}_s \]  \hspace{1cm} (5.124)

The various synchronous machine models suitable for stability studies follow directly from equations (5.123) and (5.124). A few examples are given below.

**Model 1 -- One damper winding on each axis**

With one damper winding on each of the \( d \) and \( q \) axes we have

\[
\begin{align*}
\Psi_r &= \begin{bmatrix} \psi_{rd} \\ \psi_{ld} \\ \psi_{1q} \end{bmatrix}, \\
V_r &= \begin{bmatrix} e_{rd} \\ 0 \\ 0 \end{bmatrix}, \\
V_s &= \begin{bmatrix} e_d \\ e_q \end{bmatrix}, \\
I_s &= \begin{bmatrix} i_d \\ i_q \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{X}_{rr} &= \begin{bmatrix} x_{fdd} & x_{fld} \\ x_{fld} & x_{11d} \end{bmatrix}, \\
\mathbf{X}_{sr} &= \begin{bmatrix} x_{gfd} & x_{a1d} \\ x_{a1l} & x_{a1q} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{R}_r &= \begin{bmatrix} -r_{fd} \\ -r_{ld} \\ -r_{1q} \end{bmatrix}, \\
\mathbf{X}_{ss} &= \begin{bmatrix} -x_d \\ -x_q \end{bmatrix}, \\
\mathbf{R}_s &= \begin{bmatrix} -r \\ -r \end{bmatrix}
\end{align*}
\]

Therefore

\[
\mathbf{X}_{rr}^{-1} = \begin{bmatrix} A x_{11d} & -A x_{f1d} \\ -A x_{f1d} & A x_{fdd} \end{bmatrix} \begin{bmatrix} 1/x_{11q} \end{bmatrix}
\]

where

\[
A = 1/(x_{11d} x_{fdd} - x_{f1d}^2)
\]

\[
\mathbf{X}_{sr} \mathbf{X}_{rr}^{-1} = \begin{bmatrix} A(x_{11d} x_{a1d} - x_{f1d} x_{a1d}) & A(x_{fdd} x_{a1d} - x_{f1d} x_{a1d}) \\ x_{a1l} & x_{a1q} \end{bmatrix}
\]

Assuming \( \omega = \omega_o \)
SYNCHRONOUS MACHINES

\[
WX_{sr}X_{rr}^{-1} = \begin{bmatrix}
A(x_{1ld}x_{afd} - x_{f1d}x_{ald}) & A(x_{f1d}x_{ald} - x_{f1d}x_{afd}) \\
-x_d + A(x_{1ld}x_{afd}^2 - 2x_{f1d}x_{ald}x_{afd} + x_{f1d}x_{afd}^2)
\end{bmatrix}
\]

Similarly,

\[
W\left[X_{ss} - X_{sr}X_{rr}^{-1}X_{rs}\right] = \begin{bmatrix}
-x_d + A(x_{1ld}x_{afd}^2 - 2x_{f1d}x_{ald}x_{afd} + x_{f1d}x_{afd}^2)
\end{bmatrix}
\]

Also

\[
R_{r}X_{rr}^{-1}X_{rs} = \begin{bmatrix}
A(x_{1ld}x_{afd} - x_{f1d}x_{ald})r_{fd} & A(x_{f1d}x_{ald} - x_{f1d}x_{afd})r_{ld} \\
-x_d - A(x_{1ld}x_{afd}^2 - 2x_{f1d}x_{ald}x_{afd} + x_{f1d}x_{afd}^2)
\end{bmatrix}
\]

Therefore, from (5.123) and (5.124),

\[
\begin{bmatrix}
e_d \\ e_q
\end{bmatrix} = \begin{bmatrix}
-x_{alq} \psi_{1q} \\ A(x_{1ld}x_{afd} - x_{f1d}x_{ald})\psi_{fd} + A(x_{f1d}x_{ald} - x_{f1d}x_{afd})\psi_{1d}
\end{bmatrix} - \begin{bmatrix}
r & \left(x_q - \frac{x_{alq}}{x_{11q}}\right) \\
x_d - A(x_{1ld}x_{afd}^2 - 2x_{f1d}x_{ald}x_{afd} + x_{f1d}x_{afd}^2) & r
\end{bmatrix} \begin{bmatrix}
i_d \\ i_q
\end{bmatrix}
\]

and

\[
\frac{1}{\omega_o} \begin{bmatrix}
\psi_{fd} \\ \psi_{1d} \\ \psi_{ld} \\ \psi_{1q}
\end{bmatrix} = \begin{bmatrix}
e_f \\ 0 \\ 0 
\end{bmatrix} + \begin{bmatrix}
A_{r_fd} \left(-x_{1ld}\psi_{fd} + x_{f1d}\psi_{1d}\right) & A_{r_{fd}}\left(x_{f1d}\psi_{fd} - x_{f1d}\psi_{1d}\right) \\
A_{r_{ld}} \left(x_{f1d}\psi_{ld} - x_{f1d}\psi_{1d}\right) & A_{r_{ld}} \left(x_{f1d}\psi_{ld} - x_{f1d}\psi_{1d}\right)
\end{bmatrix} \begin{bmatrix}
i_d \\ i_q
\end{bmatrix}
\]

Equation (5.125) can also be written as
\[
\begin{bmatrix}
e_d \\
e_q
\end{bmatrix} = \begin{bmatrix}
e_d' \\
e_q'
\end{bmatrix} - \begin{bmatrix}
r & -x_{q}'' \\
x_{d}'' & r
\end{bmatrix} \begin{bmatrix}
i_d \\
i_q'
\end{bmatrix}
\]

(5.127)

where \(x_{d}''\) and \(x_{q}''\) are the \(d\) and \(q\) axis subtransient reactances given by equations (5.91) and (5.99). 
\(e_d''\) and \(e_q''\) are fictitious voltages behind subtransient reactances. These are proportional to the various flux linkages as indicated in equation (5.125).

Although equations (5.125) and (5.126) are quite straightforward to implement, often the machine parameter values used in these equations are not available. Therefore, it may be desirable to express these in terms of the constants supplied by the manufacturers and as defined earlier. We will use the relationships in (5.57).

From equations (5.92) and (5.96)

\[
x_d'' - x_i = \frac{x_{ad} x_{fd} x_{1d}}{x_{ad} (x_{1d} + x_{fd}) + x_{1d} x_{fd}}
\]

(5.128)

\[
x_d' - x_i = \frac{x_{ad} x_{fd}}{x_{fd}}
\]

(5.129)

Also

\[
x_d - x_d' = \frac{x_{ad}^2}{x_{fd}}
\]

(5.130)

Dividing (5.129) by (5.130)

\[
x_{fd} = \frac{x_d' - x_i}{x_d - x_d'} x_{ad}
\]

(5.131)

Subtracting (5.128) from (5.129)

\[
x_d' - x_d'' = \frac{(x_d' - x_i) x_{ad} x_{fd}}{x_{ad} (x_{1d} + x_{fd}) + x_{1d} x_{fd}}
\]

(5.132)

Dividing (5.128) by (5.132)

\[
x_{1d} = \frac{(x_d' - x_i) (x_d'' - x_i)}{x_d' - x_d''}
\]

(5.133)

The first of the set of equations (5.126) can be written as, after multiplying both sides by \(\frac{x_{ad}}{r_{fd}}\),

\[
\frac{x_{ad}}{\omega_r r_{fd}} \frac{d}{dt} \psi_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd} - A x_{ad} (x_{11d} \psi_{fd} - x_{f1d} \psi_{1d}) - A x_{ad} (x_{11d} x_{ad} - x_{f1d} x_{ad}) i_d
\]

Defining \(e_q' = \frac{x_{ad}}{x_{fd}} \psi_{fd}\), and noting that \(\frac{x_{ad}}{r_{fd}} e_{fd} = E_{fd}\),

\[
\frac{x_{fd}}{\omega_r r_{fd}} \frac{d}{dt} e_q' = E_{fd} - A x_{11d} x_{fd} e_q' + A x_{ad}^2 \psi_{1d} - A x_{ad} x_{1d} i_d
\]

\[= E_{fd} - (1 + A x_{ad}^2) e_q' + A x_{ad}^2 \psi_{1d} - A x_{ad} x_{1d} i_d\]

(5.134)
Noting that 
\[ A = \frac{1}{x_{11d} x_{f1d} - x_{ad}^2} = \frac{1}{x_{ad} (x_{f1d} + x_{id}) + x_{f1d} x_{id}} \]
\[ A x_{ad}^2 = \frac{x_{ad}^2}{x_{ad} (x_{f1d} + x_{id}) + x_{f1d} x_{id}} = \frac{x_{d}^2 - x_{d}^2}{x_{d}^2} \]
\[ x_{ad} = \frac{(x_{d} - x_{d}^2)(x_{d} - x_{d}^2)}{(x_{d} - x_{d})^2} \]

(using (5.132) and (5.131))

Using the expression for \( x_{1d} \) from (5.133), (5.134) can therefore be written as

\[ T'_{do} \frac{d}{dt} e'_q = E_{f1d} - \left[ 1 + \frac{(x_{d} - x_{d}^2)(x_{d} - x_{d}^2)}{(x_{d} - x_{d})^2} \right] e'_q \]
\[ + \frac{(x_{d} - x_{d}^2)(x_{d} - x_{d}^2)}{(x_{d} - x_{d})^2} \psi_{id} - \frac{(x_{d} - x_{d}^2)(x_{d} - x_{d})}{x_{d} - x_{d}} i_d \]

(5.135)

The second of the set of equations (5.126)

\[ \frac{1}{\omega_o} \frac{d}{dt} \psi_{id} = A r_{id} \left[ (x_{f1d} \psi_{f1d} - x_{f1d} \psi_{id}) - (x_{f1d} x_{alid} - x_{f1d} x_{f1d}) i_d \right] \]

(5.136)

\[ A r_{id} = \frac{r_{id}}{x_{f1d}} = \frac{1}{x_{f1d} \omega_o T_{do}^\prime} \]

Therefore, (5.136) can be written as

\[ T_{do}^\prime \frac{d}{dt} \psi_{id} = \frac{x_{ad}}{x_{f1d}} \psi_{f1d} - \psi_{id} - \frac{x_{f1d} x_{f1d}}{x_{f1d}} i_d \]
\[ = e'_q - \psi_{id} - (x_{d} - x_{d}) i_d \]

(5.137)

The \( q \) axis equation is, from (5.126),

\[ \frac{1}{\omega_o} \frac{d}{dt} \psi_{iq} = - \frac{r_{iq}}{x_{11q}} \psi_{iq} - \frac{x_{alq}}{x_{11q}} r_{iq} i_q \]

Multiplying both sides by \( \frac{x_{11q}}{r_{iq}} \)

\[ \frac{x_{11q}}{\omega_o r_{iq}} \frac{d}{dt} \psi_{iq} = - \psi_{iq} - x_{aq} i_q \]

With \( e_{a}^\star = - \frac{x_{aq}}{x_{11q}} \psi_{iq} \) as defined in (5.127), and noting that \( T_{dq}^\star = \frac{x_{11q}}{\omega_o r_{iq}} \), the above can be written as
\[ T_{qo} \frac{de_q''}{dt} = (x_q - x_{d}^*) i_d - e_d'' \]  

(5.138)

\( e_q'' \) is given by, from (5.125),

\[ e_q'' = A(x_{11d} x_{afd} - x_{f1d} x_{a1d}) \psi_{jd} + A(x_{f1d} x_{afd} - x_{f1d} x_{afd}) \psi_{ld} \]

which, using (5.128), (5.129) and (5.132), reduces to

\[ e_q'' = \frac{x_{d} - x_{i}}{x_{d} - x_{i}} e_q' + \frac{x_{d}' - x_{d}}{x_{d}' - x_{i}} \psi_{ld} \]  

(5.139)

The complete model is given by equations (5.135), (5.137), (5.138), and (5.127).

The electrical output power can be expressed as

\[ P_e = e_d i_d + e_q i_q = e_d'' i_d + e_q'' i_q - (i_d^2 + i_q^2) r + (x_q'' - x_{d}'') i_d i_q \]  

(5.140)

**Initial conditions**

From initial power flow, terminal voltage and current (\( e \) and \( i \)), and the power factor angle, \( \phi \), will be known. From the phasor diagram of Figure 5.2,

\[ \delta_o = \tan^{-1} \frac{x_q i \cos \phi - r i \sin \phi}{e + x_q i \sin \phi + r i \cos \phi} \]

\[ e_{do} = e \sin \delta_o, \quad e_{qo} = e \cos \delta_o \]

\[ i_{do} = i \sin(\delta_o - \phi), \quad i_{qo} = i \cos(\delta_o - \phi) \]

(Note: \( \phi \) is negative for lagging current)

\[ E_{fd} = e_{qo} + r i_{qo} + x_d i_{do} \]

In the steady state the left hand sides of equations (5.135), (5.137) and (5.138) are zero. Therefore, from (5.137)

\[ \psi_{ido} = e_{qo}' - (x_d' - x_i) i_{do} \]  

(5.141)

Substituting \( \psi_{ido} \) in (5.135) and simplifying, \( e_{qo}' \) is obtained as

\[ e_{qo}' = E_{fd}' - (x_d' - x_{d}') i_{do} \]  

(5.142)

From (5.138)

\[ e_{do}'' = (x_q - x_{q}'') i_{qo} \]  

(5.143)

During the step-by-step computation \( e_{q}'' \) for equation (5.127) is obtained from equation (5.139).

An alternative representation of this model using voltages behind transient and subtransient impedances as variables in the differential equations is as follows[9, 10]:

\[ T_{do}' \frac{de_q'}{dt} = E_{fd} - (x_d' - x_{d}') i_d - e_q' \]  

(5.144)
along with equation (5.127). The derivation is given at the end of this chapter.

**Model 2 -- One equivalent damper winding on the q axis only**

With one equivalent damper winding on the q axis only,

\[
\begin{align*}
\Psi_r &= \begin{bmatrix} \psi_{fd} \\ \psi_{1q} \end{bmatrix}, \quad V_r = \begin{bmatrix} e_{fd} \\ 0 \end{bmatrix} \\
X_{rr} &= \begin{bmatrix} x_{fd} \\ x_{11q} \end{bmatrix}, \quad X_{sr} = \begin{bmatrix} x_{ad} \\ x_{aq} \end{bmatrix}, \quad R_r = \begin{bmatrix} -r_{fd} \\ -r_{1q} \end{bmatrix}
\end{align*}
\]

All other submatrices are as in model 1.

Following the same steps as before,

\[
W X_r X_r^{-1} = \begin{bmatrix}
-x_{aq} & \frac{X_{aq}^2}{X_{11q}} \\
-x_{ad} & \frac{x_{ad}^2}{x_{fd}} \\
x_{fd} & -x_{11q}
\end{bmatrix}
\]

\[
W \left[ X_{ss} - X_{sr} X_r^{-1} X_{rs} \right] = \begin{bmatrix}
X_q & \frac{x_{aq}^2}{x_{11q}} \\
-x_d + \frac{x_{ad}^2}{x_{fd}} & x_{11q}
\end{bmatrix}
\]

\[
R_r X_r^{-1} = \begin{bmatrix}
-r_{fd} & \frac{r_{1q}}{x_{11q}} \\
x_{fd} & -x_{11q}
\end{bmatrix}
\]

and

\[
R_r X_{rs} X_r = \begin{bmatrix}
\frac{x_{ad} r_{fd}}{x_{fd}} & x_{aq} r_{1q} \\
x_{fd} & \frac{x_{aq} r_{1q}}{x_{11q}}
\end{bmatrix}
\]

Therefore, from equations (5.123) and (5.124)
Defining the following quantities

\[ e'_d = -\frac{x_{aq}}{x_{11q}} \psi_{1q} \quad e'_q = \frac{x_{ad}}{x_{fjd}} \psi_{fd} \]

and using the expressions for transient reactances and time constants, equations (5.147) and (5.148) can be expressed as

\[
\begin{bmatrix}
    e'_d \\
    e'_q
\end{bmatrix} = \begin{bmatrix}
    e'_d \\
    e'_q
\end{bmatrix} - \begin{bmatrix}
    r & -x'_q \\
    x'_d & r
\end{bmatrix} \begin{bmatrix}
    i_d \\
    i_q
\end{bmatrix}
\]

(5.149)

\[
T'_{do} \frac{d}{dt} e'_q = E_{fd} - (x'_d - x'_q) i_d - e'_q
\]

(5.150)

\[
T'_{qo} \frac{d}{dt} e'_d = (x_q - x'_q) i_q - e'_d
\]

Note that in the above equations, transient notations have been used for the \( q \) axis quantities so as to be consistent with the notations for the \( d \) axis.

**Model 3 -- no damper winding**

Following the same procedure as before, the model can be derived as

\[
\begin{bmatrix}
    e'_d \\
    e'_q
\end{bmatrix} = \begin{bmatrix}
    0 \\
    e'_q
\end{bmatrix} - \begin{bmatrix}
    r & -x'_q \\
    x'_d & r
\end{bmatrix} \begin{bmatrix}
    i_d \\
    i_q
\end{bmatrix}
\]

(5.151)

\[
T'_{do} \frac{d}{dt} e'_q = E_{fd} - (x'_d - x'_q) i_d - e'_q
\]

(5.152)

**Saturation**

An exact analysis of the effect of saturation in a synchronous machine is complex. The assumption is usually made that saturation is a function of the air-gap flux. A typical no-load saturation curve is shown in Figure 5.3. The extension of the straight line portion of the curve is known as the air-gap line.
Fig. 5.3 A typical no-load saturation curve of a synchronous machine.

It can be seen that the presence of saturation requires an additional field current (or mmf), \( \Delta I_f = I_f - I_f' \), over that required to generate the voltage \( E_a \) on the air-gap line. A no-load saturation factor may be defined as

\[
S = \frac{\Delta I_f}{I_f} = \frac{E_s}{E_a}
\]  
(5.153)

Denoting the saturated value of \( x_{ad} \) by \( x_{adS} \), we can write

\[
x_{adS} I_f = x_{ad} I_f'
\]

or

\[
\frac{x_{adS}}{x_{ad}} = \frac{1}{1 + S}
\]  
(5.154)

**Saturation in cylindrical rotor machines**

The assumption is made that \( x_d = x_q, x_{ad} = x_{aq} \), and that at a given field excitation, saturation is a function of the total air-gap flux. Therefore, in the steady state, analogous to the no-load saturation relationships (5.153), it can be stated that the phasor difference between the voltage generated by the actual field current and the armature reaction drop \( (x_d - x_l)I \) is \( (1+S) \) times the voltage behind the leakage reactance drop. The steady state phasor relationship including the effect of saturation is shown in Figure 5.4.
S is a function of the total air-gap flux ($E_a$), and the field excitation ($E$). At constant field current, $S$ is a function of only the total air-gap flux. Evaluation of $S$ under various loading conditions, therefore, requires the complete load saturation characteristics which are usually not available.

The use of Potier reactance (or the transient reactance), which is generally greater than the calculated leakage reactance, provides an empirical correction of the saturation factor obtained from the no-load saturation curve to allow for load saturation. Potier reactance can be determined from test as illustrated in Figure 5.5.

Saturation in salient pole machines

In the case of salient pole machines, although the stator core and teeth are saturated to the same extent in both the direct and quadrature axes, there is generally very little saturation in the quadrature axis magnetic path in the rotor. Most of the saturation in the rotor exists in the body of the pole, which is in the path of the direct axis flux. Therefore, the assumption is made that the mmf required to overcome saturation can be divided into two components: one which is dependent only on the total flux, the other dependent only on the direct axis flux. Denoting the two saturation factors by $S_1$ and $S_2$, the phasor diagram shown in Figure 5.6 can be constructed.
In practice, for both salient pole and cylindrical rotor machines, a single saturation factor, $S_D$, can be used to account for saturation on the direct axis. This may be obtained from the no-load saturation curve using the Potier (or transient) reactance (5.153). The saturation factor, $S_Q$, for the quadrature axis may be taken as

$$S_Q = \frac{x_q}{x_d} S_D$$

### Inclusion of the effect of saturation in machine modeling

From the previous discussion it is apparent that saturation predominantly affects the mutual reactances, $x_{ad}$, $x_{aq}$, although some of the rotor leakage reactances also saturate to some extent. Therefore, in detailed machine modeling, where the equations are expressed explicitly in terms of the mutual and leakage reactances, saturation can be accounted for by simply adjusting the values of $x_{ad}$ and $x_{aq}$ during the step-by-step computation, according to

$$x_{adS} = \frac{x_{ad}}{1 + S_D}$$

$$x_{aqS} = \frac{x_{aq}}{1 + S_Q}$$

$S_D$ and $S_Q$ can be computed as described earlier. In a stability program it is more convenient to use an analytic function defining the saturation factor at various points on the saturation curve. One possible function is

$$S_D = 0, \text{ for } E_p \leq A$$

$$S_D = \frac{B(E_p - A)^2}{E_p}, \text{ for } E_p > A$$

where $A$ and $B$ are constants which can be calculated by solving the equations

$$S_{D1.0} = B(1.0 - A)^2 / 1.0$$

$$S_{D1.2} = B(1.2 - A)^2 / 1.2$$

$$S_{D1.0} = B(1.0 - A)^2 / 1.0$$

$$S_{D1.2} = B(1.2 - A)^2 / 1.2$$

5-35
$S_{D\,1.0}$ and $S_{D\,1.2}$ are the values of $S_D$ at two points, $E_p = 1.0$ and $E_p = 1.2$, on the no-load saturation curve (see Figure 5.3).

Changing the values of $x_{ad}$ and $x_{pq}$ continuously during the stability computation can be cumbersome and time consuming. In less detailed machine models, it is desirable to use fixed reactance values in the machine equations, and account for saturation using separate terms. This can be accomplished as shown below.

Consider the model with one rotor circuit on each axis (field winding on the direct axis and one amortisseur winding on the quadrature axis), as discussed under model 2. Rewriting the starting point equations, using saturated values for the mutual reactances,

\[
\psi_{fd} = (x_{ads} + x_{fs})i_{fd} - x_{ads}i_d
\]  
(5.160)

\[
\psi_d = -(x_{ads} + x_i)i_d + x_{ads}i_{fd}
\]  
(5.161)

\[
\psi_{1q} = (x_{aq} + x_i)i_{1q} - x_{aq}i_q
\]  
(5.162)

\[
\psi_q = -(x_{aq} + x_i)i_q + x_{aq}i_{1q}
\]  
(5.163)

\[
e_{fd} = \frac{1}{\omega_o} \frac{d}{dt} \psi_{fd} + r_{fd}i_{fd}
\]  
(5.164)

\[
e_d = -\psi_q - ri_d
\]  
(5.165)

\[
e_q = \psi_d - ri_q
\]  
(5.166)

From (5.160)

\[
\psi_{fd} = x_{ads}(i_{fd} - i_d) + x_{fs}i_{fd}
\]

\[
= x_{ad}(i_{fd} - i_d) - Sx_{ads}(i_{fd} - i_d) + x_{fs}i_{fd}
\]

(assuming (5.154))

\[
\approx x_{ad}(i_{fd} - i_d) - SE_{pq} + x_{fs}i_{fd}
\]

\[
\approx x_{fdd}i_{fd} - x_{ad}i_d - S(1 + \frac{x_f}{x_{ad}})E_{pq}
\]

(assuming that the ratio of the saturation of the field leakage reactance to the saturation of the mutual reactance is in proportion to the ratio of the field leakage reactance to the mutual reactance)

\[
\therefore e'_{q} = \frac{x_{ad}}{x_{fdd}} \psi_{fd} = x_{ad}i_{fd} - \frac{x_{ad}}{x_{fdd}}i_d - SE_{pq} = x_{ad}i_{fd} - (x_d - x'_d)i_d - SE_{pq}
\]  
(5.167)

or

\[
e'_q = x_{ad}i_{fd} - (x_d - x'_d)i_d - S' e'_q
\]  
(5.168)

From (5.161) and (5.166)
\[ e_q = x_{ad} (i_{fd} - i_d) - x_i i_d - r i_q \]
\[ = x_{ad} (i_{fd} - i_d) - S x_{ad} (i_{fd} - i_d) - x_i i_d - r i_q \]
\[ = x_{ad} i_{fd} - x_d i_d - S E_{pq} - r i_q \]  \hspace{1cm} \text{(5.169)}

From (5.167) and (5.169)
\[ e_q = e'_q - x'_d i_d - r i_q \]  \hspace{1cm} \text{(5.170)}

From (5.164)
\[ \frac{1}{\omega_o} \frac{d}{dt} \left( \frac{x_{ad}}{x_{fjd}} \psi_{fd} \right) = \frac{x_{ad}}{x_{fjd}} e_{fd} - \frac{x_{ad}}{x_{fjd}} r_{fd} i_{fd} \]
or
\[ \frac{x_{fjd}}{\omega_o} r_{fd} \frac{d}{dt} e'_q = x_{ad} \frac{e_{fd}}{r_{fd}} - x_{ad} i_{fd} = E_{fd} - x_{ad} i_{fd} \]  \hspace{1cm} \text{(5.171)}

From (5.168) and (5.171), we can write
\[ T_{do} \frac{d}{dt} e'_q = E_{fd} - (x_d - x'_d) i_d - (1 + S) e'_q \]  \hspace{1cm} \text{(5.172)}

This equation is of the same form as derived previously neglecting saturation, except for the additional term \(-S e'_q\) on the right-hand side, which accounts for saturation.

The equations for the \(d\) axis can be derived similarly. The complete model including saturation effect is therefore as follows:
\[ \begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} e'_d \\ e'_q \end{bmatrix} - \begin{bmatrix} r & -x'_d \\ x'_d & r \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} \]  \hspace{1cm} \text{(5.173)}

\[ T_{do} \frac{d}{dt} e'_q = E_{fd} - (x_d - x'_d) i_d - (1 + S) e'_q \]  \hspace{1cm} \text{(5.174)}

\[ T_{do} \frac{d}{dt} e'_d = (x_q - x'_q) i_q - (1 + S) e'_d \]  \hspace{1cm} \text{(5.175)}

**Network Model in DQ Reference Frame**

Consider a line between two buses, numbered 1 and 2, in a network. The voltage and current relationship is
\[ \mathbf{v}_{1,abc} = r_{12} \mathbf{i}_{abc} + l_{12} \frac{d}{dt} \mathbf{i}_{abc} + \mathbf{v}_{2,abc} \]  \hspace{1cm} \text{(5.176)}

where \(r_{12}\) and \(l_{12}\) are the resistance and inductance of the line.

Resolving the phase quantities along a set of synchronously rotating network \(D-Q\) axes using the transformations (5.15) and (5.17),
\[ \mathbf{e}_{1,DQQ} = r_{12} \mathbf{i}_{DQQ} + l_{12} \frac{d}{dt} (T^{-1} \mathbf{i}_{DQQ}) + \mathbf{e}_{2,DQQ} \]
After carrying out the indicated matrix operation
\[ e_{1,\text{DQO}} = r_{12}i_{\text{DQO}} + l_{12}\omega S i_{\text{DQO}} + l_{12}\frac{d}{dt}i_{\text{DQO}} + e_{2,\text{DQO}} \]  
(5.177)
where
\[ S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

In the steady state \( \frac{d}{dt}i_{\text{DQO}} \) is zero. Even in the transient state this term is small compared to the \( \omega i_{\text{DQO}} \) term, and in conventional stability studies it can be neglected. The \( \frac{d}{dt}i_{\text{DQO}} \) term accounts for the short-lived electrical transients that are present immediately after a disturbance. Therefore, for balanced operation (\( e_0 = i_0 = 0 \)), we can write (5.177) as
\[ e_{1D} = r_{12}i_D - \omega l_{12}i_Q + e_{2D} \]
\[ e_{1Q} = r_{12}i_Q + \omega l_{12}i_D + e_{2Q} \]
which, in per unit, become
\[ e_{1D} = r_{12}i_D - x_{12}i_Q + e_{2D} \]
\[ e_{1Q} = r_{12}i_Q + x_{12}i_D + e_{2Q} \]
(5.178)
Defining complex voltage and current
\[ \hat{e} = e_D + je_Q, \quad \hat{i} = i_D + ji_Q \]
(5.178) can be expressed as
\[ \hat{e}_1 = (r_{12} + jx_{12})\hat{i} + \hat{e}_2 \]
or
\[ \hat{i} = \frac{\hat{e}_1 - \hat{e}_2}{z_{12}} = y_{12}(\hat{e}_1 - \hat{e}_2) \]
(5.179)
Thus the relationship between current and voltage in \( DQ \) reference is the same as that when represented by conventional phasors. Therefore, for balanced operation, network equations using the network impedance or admittance matrix as discussed in Chapter 1 will remain the same in the \( DQ \) reference.

**Transformation from machine to network reference frame**

In the representation of synchronous machines the machine quantities are expressed in terms of direct and quadrature axes that are oriented differently for different machines. For a multi-machine stability analysis, it is therefore necessary to transform the machine currents and voltages from the individual \( d-q \) axes (machine reference frame) to a common set of network \( D-Q \) axes (network reference frame), before the network solution can be undertaken. Following the network solution the quantities can then be transformed back to the machine reference frame for use in the solution of the machine equations.
The principle of transformation is illustrated in Figure 5.7. \( \delta \) is the angle between a machine \( q \) axis and the network \( D \) axis. In order to transform from one set of axes to another, we rotate the \( d-q \) axes along with the particular phasor (voltage or current) by an angle of \( \pi/2 - \delta \), so as to make the two sets of axes coincide. In terms of the network reference frame the rotation can therefore be expressed as

\[
(D + jQ) e^{j(\pi/2 - \delta)} = d + jq
\]

or

\[
(D + jQ)(\sin \delta + j \cos \delta) = d + jq
\]

Separating the real and imaginary parts, the transformation can be expressed in matrix form as

\[
\begin{bmatrix}
d \\
q
\end{bmatrix} =
\begin{bmatrix}
\sin \delta & -\cos \delta \\
\cos \delta & \sin \delta
\end{bmatrix}
\begin{bmatrix}
D \\
Q
\end{bmatrix}
\]

The reverse transformation can be expressed as

\[
\begin{bmatrix}
D \\
Q
\end{bmatrix} =
\begin{bmatrix}
\sin \delta & \cos \delta \\
-\cos \delta & \sin \delta
\end{bmatrix}
\begin{bmatrix}
d \\
q
\end{bmatrix}
\]

**Stability Computation in Multi-Machine Systems**

For the purpose of this illustration and to keep the analysis simple, we will assume that the network has been reduced retaining only the nodes at which machines are connected. As has been pointed out in Chapters 3 and 4, working with the full network has computational advantage as the network admittance matrix is generally very sparse. The modifications in the computational steps that are required when working with the full network will be similar to those using the classical machine model as discussed in Chapters 3 and 4.

For \( n \) machines, the current and voltage relationships in machine and network reference frame can be written as, from (5.182),

\[
i = TI
\]

\[
e = TV
\]
where \( \mathbf{i} (\mathbf{v}) \) and \( \mathbf{I} (\mathbf{V}) \) are the vectors of the currents (voltages) in machine and network reference frame, respectively.

\[
\mathbf{i} = \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \\ \vdots \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \\ \vdots \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} e_{d1} \\ e_{q1} \\ e_{d2} \\ e_{q2} \\ \vdots \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} e_{d1} \\ e_{q1} \\ e_{d2} \\ e_{q2} \\ \vdots \end{bmatrix}
\]

and

\[
\mathbf{T} = \begin{bmatrix} \sin \delta_1 & -\cos \delta_1 \\ \cos \delta_1 & \sin \delta_1 \\ & & \sin \delta_2 & -\cos \delta_2 \\ & & \cos \delta_2 & \sin \delta_2 \\ & & & & \ddots \end{bmatrix}
\]

Using machine model 1, for the purpose of illustration, the relationships between machine internal and terminal voltages can be written, from equation (5.127), for \( n \) machines, as

\[
\mathbf{E} = \mathbf{Z}_M \mathbf{i} + \mathbf{e} \quad \text{(5.185)}
\]

where

\[
\mathbf{E} = \begin{bmatrix} e_{d1}^* \\ e_{q1}^* \\ e_{d2}^* \\ e_{q2}^* \\ \vdots \end{bmatrix}, \quad \mathbf{Z}_M = \begin{bmatrix} r_1 & -x_{q1}^* \\ x_{d1}^* & r_1 \\ & & r_2 & -x_{q2}^* \\ & & x_{d2}^* & r_2 \\ & & & & \ddots \end{bmatrix}
\]

The network equations, after separating the real and imaginary parts, can be written in real form as

\[
\mathbf{I} = \mathbf{Y}_N \mathbf{V} \quad \text{(5.186)}
\]

where

\[
\mathbf{Y}_N = \begin{bmatrix} g_{11} & -b_{11} & g_{12} & -b_{12} & \cdots \\ b_{11} & g_{11} & b_{12} & g_{12} & \cdots \\ g_{21} & -b_{21} & \cdots \\ b_{21} & g_{21} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}
\]

From (5.185)

\[
\mathbf{i} = \mathbf{Z}_M^{-1} \mathbf{E} - \mathbf{Z}_M^{-1} \mathbf{e}
\]

\[
\therefore \mathbf{I} = \mathbf{T}^{-1} \mathbf{i} = \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{E} - \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{e} = \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{E} - \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{T} \mathbf{V} \quad \text{(5.187)}
\]

From (5.186) and (5.187)

\[
\mathbf{V} = \left[ \mathbf{Y}_N + \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{T} \right]^{-1} \mathbf{T}^{-1} \mathbf{Z}_M^{-1} \mathbf{E}
\]

5-40
\[
\dot{e} = TV = T[Y_N + T^{-1}Z_M^{-1}T]^{-1}T^{-1}Z_M^{-1}E
\]
(5.188)
The machine terminal currents are obtained from (5.185)
\[
i = Z_M^{-1}[E - e]
\]
(5.189)
Electrical power output is obtained from (5.140).
The computational steps shown in (5.188) and (5.189) are equivalent to those described in Chapter 4 (see equations (4.27), (4.28)), where the machine internal voltage in series with the internal impedance is first converted to a current source and machine terminal voltage is obtained directly by multiplying the vector composed of these currents by the inverse of the modified network admittance matrix. In (5.188), the operation \(Z_M^{-1}E\) is equivalent to converting the voltage source into current source in machine reference frame. This is then transformed to network reference frame, followed by multiplication by the inverse of the modified network admittance matrix to obtain the machine terminal voltage. Finally, this is transformed back to the machine reference frame. The machine current is then obtained from (5.189).

Note that in using machine model 1, since \(x''_d \approx x''_q\), they can be assumed equal and then \(T^{-1}Z_M^{-1}T = Z_M^{-1}\). Thus \([Y_N + T^{-1}Z_M^{-1}T]\) has to be inverted only once for each network configuration. Also, with \(x''_d = x''_q\), it is possible to work with equations in complex form thereby reducing the computational burden considerably. Equation (5.188) would then be written as
\[
\hat{e} = \bar{T}[\overline{V}_N + \overline{Z}_M^{-1}]^{-1}\overline{T}^{-1}\overline{Z}_M^{-1}\hat{E}
\]
(5.190)
where \(\hat{e}\) and \(\hat{E}\) are the vectors of complex machine terminal and internal voltages, respectively. \(\overline{V}_N\) is the complex network admittance matrix. \(\overline{Z}_M\) is a diagonal matrix of the complex machine internal impedances \(r_i + jx''_{di}\), and \(\bar{T}\) is given by
\[
\bar{T} = \begin{bmatrix}
j\left(\frac{\pi}{2} - \delta_i\right) & \cdots & \cdots \\
E & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{bmatrix}
\]
Equation (5.189) would then become
\[
\hat{i} = Z_M^{-1}[\hat{E} - \hat{e}]
\]
(5.191)
Note that in this formulation the non-generator buses can be retained. As has been pointed out in Chapter 7, non-dynamic (or static) loads should be represented by constant impedance. Dynamic loads (motor and non-motor) can be adequately represented by one of the induction machine models discussed in Chapter 7. For the buses with impedance loads, the injected currents would be zero, the block diagonal elements of \(Z_M\) would be \(2 \times 2\) null matrix, the impedance having been included in the network admittance matrix, and the corresponding elements of the \(T\) matrix would be \(2 \times 2\) unit matrix. The network solution can then proceed as discussed in Chapter 4, taking advantage of the sparsity of \(Y_N\). Induction motor and other dynamic loads can be handled following the procedure discussed in Chapter 8.
An iterative method for handling saliency

In the previous illustration using machine model 1 it was pointed out that since $x_d'' \approx x_q''$ for most synchronous machines they can be assumed equal and network computation can be greatly simplified. For machine models 2 and 3 this assumption is not valid. However, equation (5.190) can still be used when saliency is present, using an iterative technique.

Consider machine model 2. Rewriting the machine voltage equations (5.149)

$$
e_d' = e_d + r i_d - x_d' i_q
$$

(5.192)

$$
e_q' = e_q + r i_q + x_d' i_d
$$

(5.193)

For $x_q' < x_d'$ a voltage on the $d$ axis is defined as

$$
e_{d1} = e_d + r i_d - x_d' i_q
$$

(5.194)

which is related to $e_d'$ by

$$
e_{d1} = e_d' + (x_q' - x_d') i_q
$$

(5.195)

Equations (5.193) and (5.194) can be arranged in matrix form as

$$
\begin{bmatrix}
e_{d1} \\
e_q'
\end{bmatrix} = \begin{bmatrix}
e_d \\
e_q
\end{bmatrix} + \begin{bmatrix}
r & -x_d' \\
x_d' & r
\end{bmatrix} \begin{bmatrix}
i_d \\
i_q
\end{bmatrix}
$$

(5.196)

In (5.196) saliency has been removed. The new defined voltage $e_{d1}$ is undetermined, and this is computed in an iterative way. The complete process in a multi-machine system can be summarized as follows:

The matrix $\left[ Y_N + Z_M^{-1} \right]^{-1}$ is formed only once for a particular network condition. At each integration step $e_d'$ and $e_q'$ are known. $e_{d1}$ is determined for each machine from (5.195) using the latest value of $i_q$. This voltage and $e_q'$ are then used in the voltage vector in (5.188). New values of $i_q$ and $e_{d1}$ are computed for each machine from (5.188) and (5.195), respectively. The voltage vector in (5.188) is then updated. The process is repeated until convergence is obtained.

The convergence is reasonably fast except immediately following a major disturbance. It can be made faster by using some acceleration technique.

It is found that convergence is not achieved if in (5.195) $x_q'$ is greater than $x_d'$. When $x_q' > x_d'$ (for example, using machine model 3 when $x_q$ replaces $x_q'$ and $x_q > x_d'$), a voltage on the $q$ axis needs to be defined as follows, in order to achieve convergence.

$$
e_{q1} = e_q + r i_q + x_d i_d
$$

(5.197)

and this is related to $e_q'$ by

$$
e_{q1} = e_q' + (x_q - x_q') i_d
$$

(5.198)

The matrix equation corresponding to (5.196) is now
A similar procedure as before can now be followed.

**Generator Capability Curve**

Consider a round rotor machine connected to an infinite system as shown in Figure 5.8. The expressions for real and reactive powers supplied to the system, neglecting resistance, are given by

\[
\begin{align*}
\delta & = \sin^{-1}\left(\frac{V P}{x_s}\right) \quad (5.200) \\
\delta & = \cos^{-1}\left(\frac{V Q + V^2}{x_s}\right) \quad (5.201)
\end{align*}
\]

From (5.200) and (5.201) we obtain

\[
P^2 + \left(\frac{Q + \frac{V^2}{x_s}}{x_s}\right)^2 = \left(\frac{EV}{x_s}\right)^2 \quad (5.202)
\]

Equation (5.202) describes a circle with radius \(EV/x_s\) and center located at \((0, -V^2/x_s)\), using \(P\) and \(Q\) as the axes of coordinates, as shown in Figure 5.9. For a given value of \(V\) we can draw a series of circles corresponding to various values of the generator excitation voltage \(E\).

The generator must be operated within the limits imposed by the stator and rotor heating, the maximum and minimum allowable turbine outputs, and stability. The various limits are shown on Figure 5.9. The stator limit is indicated by the circle of radius equal to the MVA rating with center at the origin \((0, 0)\). The rotor limit is indicated by the circle of radius \(E_{max}V/x_s\) with center located at \((0, -V^2/x_s)\).

The theoretical steady state stability limit \((\delta = 90^\circ)\) is indicated by the line parallel to the \(P\) axis and at a distance \(-V^2/x_s\) from it. This can be extended, as indicated in Figure 5.9, by the action of automatic voltage regulator. It is customary to allow for a margin (10-15%) in the steady state stability limit. The stability limit allowing for margin is obtained by first finding the power limit for a given excitation. This is then reduced by the desired margin and the corresponding point on the excitation curve is noted, as illustrated in the figure.

The complete operating region is shown by the shaded area. In practice some restriction may apply due to transient stability or other considerations.
Stability Limit at Constant Field Voltage

Consider the single machine-infinite bus system shown in Figure 5.10. Using the synchronous machine model 3 and neglecting armature and line resistances, we have

\[ T_{do} \frac{de'_q}{dt} = E_{jd} - (x_d - x'_d) i_d - e'_q \]  
\[ \frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P_m - P_e \]  
\[ P_e = e'_q i_q + (x_q - x'_d) i_q i_d \]  
\[ e_d = x_q i_q \]  
\[ e_q = e'_q - x'_d i_d \]

We also have

\[ e_d = V_b \sin \delta - x_c i_q \]  
\[ e_q = V_b \cos \delta + x_c i_d \]

From (5.207) and (5.209), we have

\[ i_d = \frac{e'_q}{x'_d + x_c} - \frac{V_b \cos \delta}{x'_d + x_c} \]

Similarly, from (5.206) and (5.208), we have

\[ i_q = \frac{V_b \sin \delta}{x_q + x_c} \]
Fig. 5.10 Phasor diagram of a synchronous machine connected to infinite bus through a reactance.

Linearizing (5.203) - (5.205), (5.210) and (5.211), and noting that $\Delta E_{fd}=0$, we have

$$T_{do}' \frac{d\Delta e_q'}{dt} = -(x_d - x'_d) \Delta i_d - \Delta e_q'$$  \hspace{1cm} (5.212)

$$\frac{2H}{\omega_e} \frac{d^2 \Delta \delta}{dt^2} = - \Delta P_c$$  \hspace{1cm} (5.213)

$$\Delta P_c = \Delta e'_q i_q + \Delta e'_q \Delta i_q + (x_q - x'_q) [\Delta i_d i_q + i_d \Delta i_q]$$  \hspace{1cm} (5.214)

$$\Delta i_d = \frac{1}{x'_d + x_e} \Delta e'_q + \frac{V_b \sin \delta}{x'_d + x_e} \Delta \delta$$  \hspace{1cm} (5.215)

$$\Delta i_q = \frac{V_b \cos \delta}{x_q + x_e} \Delta \delta$$  \hspace{1cm} (5.216)

$i_d$, $i_q$, $e_d$, $e_q$, etc. are the values at the operating point and the prefix $\Delta$ denotes the small changes in the variables about the operating point.

Substituting (5.215) and (5.216) into (5.214), we have

$$\Delta P_c = \Delta e'_q i_q + \left( e'_q + (x_q - x'_q) i_d \right) \frac{V_b \cos \delta}{x_q + x_e} \Delta \delta$$

$$+ (x_q - x'_d) i_q \left( \frac{1}{x'_d + x_e} \Delta e'_q + \frac{V_b \sin \delta}{x'_d + x_e} \Delta \delta \right)$$

which can be written as

$$\Delta P_c = a \Delta \delta + b \Delta e'_q$$  \hspace{1cm} (5.217)

where

$$a = \left( e'_q + (x_q - x'_q) i_d \right) \frac{V_b \cos \delta}{x_q + x_e} + \frac{x_q - x'_d}{x'_d + x_e} i_q V_b \sin \delta$$  \hspace{1cm} (5.218)

$$b = \frac{x_q + x_e}{x'_d + x_e} i_q$$  \hspace{1cm} (5.219)
Substituting (5.215) into (5.212), we obtain

\[
\frac{d\Delta e'_q}{dt} = -c \Delta \delta - d \Delta e'_q
\]  

(5.220)

where

\[
c = \frac{1}{T_{do}'} \frac{x_d - x'_d}{x'_d + x_e} V_b \sin \delta
\]

(5.221)

and

\[
d = \frac{1}{T_{do}'} \frac{x_d + x_e}{x'_d + x_e}
\]

(5.222)

Note that in the expressions for \( a, b, \) etc. \( e'_q, i_d, \) etc. are values at the steady-state operating point. Therefore, the expression for \( a \) can also be written as

\[
a = [E_{fd} - (x_d - x_q) i_d] V_b \cos \delta \frac{x_q - x'_d}{x'_d + x_e} + \frac{x_q - x'_d}{x'_d + x_e} i_q V_b \sin \delta
\]

(5.223)

Similarly, the expression for \( i_d \) for the initial steady-state value can also be written as

\[
i_d = \frac{E_{fd}}{x'_d + x_e} - V_b \cos \delta \frac{x_q - x'_d}{x'_d + x_e}
\]

(5.224)

Equations (5.213) and (5.220) can be written in state space form as

\[
\begin{bmatrix}
\Delta \delta \\
\Delta \dot{\omega} \\
\Delta e'_q
\end{bmatrix} =
\begin{bmatrix}
1 & \omega_o & 0 \\
\omega_o & \frac{2H}{a} & 0 \\
0 & 0 & \frac{2H}{d}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta \\
\Delta \dot{\omega} \\
\Delta e'_q
\end{bmatrix}
\]

(5.225)

The characteristic equation is

\[
\lambda^3 + d \lambda^2 + \frac{\omega_o}{2H} a \lambda + \frac{\omega_o}{2H} (ad - bc) = 0
\]

Applying Routh-Hurwitz criterion (see Appendix B), for stability, \( d > 0, a > 0, ad - bc > 0 \) and \( ad - (ad - bc) > 0 \) or \( bc > 0 \). Note that the first and fourth conditions are always satisfied since \( b > 0, c > 0 \) and \( d > 0 \). The third condition implies that \( a > 0 \).

From (5.219) and (5.221) – (5.223)

\[
ad - bc = \frac{1}{T_{do}'} \frac{x_d + x_e}{x'_d + x_e} \left[ E_{fd} - (x_d - x_q) i_d \right] V_b \cos \delta \frac{x_q - x'_d}{x'_d + x_e} + \frac{x_q - x'_d}{x'_d + x_e} i_q V_b \sin \delta
\]

Substituting the expressions for \( i_d \) and \( i_q \) from (5.224) and (5.211), the above reduces to
\[ a^2 - bc = \frac{1}{T_{do}'} \frac{E_{fd} V_b}{\delta} \cos \delta + \frac{1}{T_{do}' (x'_d + x_e)} \frac{x_d - x_q}{(x'_d + x_e)(x_q + x_e)} V_b^2 \cos 2\delta \]  
(5.226)

At stability limit \( a^2 - bc = 0 \). Therefore, from (5.226), when \( x_d = x_q \), stability limit is reached when \( \delta = 90^\circ \). For \( x_d \neq x_q \) (i.e., when saliency is present) stability limit is reached before \( \delta \) reaches 90°.

Power output at a specified \( \delta \) for constant field voltage (steady-state power) is obtained, first by rewriting (5.205) as

\[ P_e = \frac{E_{fd} V_b}{x_d + x_e} \sin \delta + \frac{1}{2} \frac{x_d - x_q}{(x'_d + x_e)(x_q + x_e)} V_b^2 \sin 2\delta \]  
(5.227)

Maximum power is reached at \( \delta < 90^\circ \).

Small disturbance stability performance can be assessed from the roots of the characteristic equation. An analytical solution for the roots is not possible. However, we can obtain the roots at the stability limit, when \( a^2 - bc = 0 \). The roots are

\[ \lambda_1 = 0, \quad \lambda_{2,3} = \frac{1}{2} \left[ -d \pm \sqrt{d^2 - \frac{2\omega_o}{H} a} \right] \]

from which we can conclude that there is some damping due to the demagnetizing effect of armature reaction, since no damping term was included in the swing equation (5.204). (Recall that with constant-voltage-behind-reactance model, with no damping term present, the response is oscillatory at constant amplitude.)

Excitation control helps counteract the demagnetizing effect of armature reaction. If we assume that due to the action of excitation control field flux linkage is maintained constant, i.e., \( \Delta e'_q \equiv 0 \), stability limit will be reached when \( a = 0 \).

Substituting for \( i_d \) and \( i_q \) from (5.210) and (5.211) into (5.218)

\[ a = \left[ e'_q + (x_q - x'_d) \frac{e'_q - V_b \cos \delta}{x'_d + x_e} \right] \frac{V_b \cos \delta}{x'_d + x_e} + \frac{x_q - x'_d}{x'_d + x_e} \frac{V_b^2 \sin^2 \delta}{x_q + x_e} \]

\[ = \frac{e'_q V_b}{x'_d + x_e} \cos \delta - \frac{x_q - x'_d}{(x'_d + x_e)(x_q + x_e)} V_b^2 \cos 2\delta \]  
(5.228)

Thus, excitation control can extend stability limit to \( \delta \) well over 90°. However, excitation control while extending the stability limit can also introduce negative damping. This is discussed in more detail in the next chapter using the concepts of synchronizing and damping torque.

Power output at a specified \( \delta \) for constant field flux linkage is obtained by substituting for \( i_d \) and \( i_q \) from (5.210) and (5.211) into (5.205). Thus

\[ P_e = \frac{e'_q V_b}{x'_d + x_e} \sin \delta - \frac{1}{2} \frac{x_q - x'_d}{(x'_d + x_e)(x_q + x_e)} V_b^2 \sin 2\delta \]  
(5.229)
Maximum power is reached at $\delta > 90^\circ$.

**Reactive power limit at $P = 0$**

Reactive power limit at $P = 0$ is obtained by setting $\delta = 0$. With $\delta = 0$,

$$ad - bc = \frac{1}{T_{do}'} \frac{E_{fd} V_b'}{x_d' + x_e'} + \frac{1}{T_{do}'} \frac{x_d' - x_q'}{(x_d' + x_e')(x_q' + x_e')} V_b'^2$$

At stability limit $ad - bc = 0$, or

$$E_{fd} + \frac{x_d'}{x_q'} V_b = 0 \quad (5.230)$$

From (5.224), with $\delta = 0$, $i_d = \frac{E_{fd} - V_b}{x_d' + x_e'}$, or

$$E_{fd} = V_b + i_d (x_d' + x_e') \quad (5.231)$$

From (5.230) and (5.231) $i_d = -\frac{V_b}{x_q' + x_e}$, or

$$Q_{\text{lim}} = \frac{V_b^2}{x_q' + x_e} \quad (\text{leading}) \quad (5.232)$$

**Self Excitation**

Under certain conditions generator rotor flux-linkages can build up to excessive levels causing severe overvoltage. This form of instability is known as self excitation. Self excitation can only occur when capacitance is present in the circuit; for example, when machines are left connected to open transmission lines or when lines are terminated by capacitors. In order to illustrate the phenomenon, consider the system shown in Figure 5.11.

**Fig. 5.11** A synchronous machine connected to a capacitive load.

Using the synchronous machine model 2, we have

$$T_{do}' \frac{d}{dt} e_q' = E_{fd} - (x_d' - x_q') i_d - e_q' \quad (5.233)$$

$$T_{do}' \frac{d}{dt} e_d' = (x_q' - x_q') i_q - e_d' \quad (5.234)$$

$$e_d = e_d' + x_q' i_q \quad (5.235)$$

$$e_q = e_q' - x_d' i_d \quad (5.236)$$
Since the machine is connected to a capacitive load,

\[ e_d = -(x_e - x_c) i_q \]  \hspace{1cm} (5.237)

\[ e_q = (x_e - x_c) i_d \]  \hspace{1cm} (5.238)

where \( x_e \) is the external reactance and \( x_c \) is the capacitive reactance.

From equations (5.236) - (5.238), we have

\[ i_d = \frac{e'_q}{x'_d + x_e - x_c} \]  \hspace{1cm} (5.239)

\[ i_q = \frac{-e'_d}{x'_q + x_e - x_c} \]  \hspace{1cm} (5.240)

Substituting the expressions for \( i_d \) and \( i_q \) into equations (5.233) and (5.234) respectively, we have

\[ T'_{do} \frac{d}{dt} e'_q = E_{fd} - \frac{x_d + x_e - x_c}{x'_d + x_e - x_c} e'_q \]  \hspace{1cm} (5.241)

and

\[ T'_{dqo} \frac{d}{dt} e'_d = -\frac{x_q + x_e - x_c}{x'_q + x_e - x_c} e'_d \]  \hspace{1cm} (5.242)

The solutions to the above equations are

\[ e'_q = \frac{E_{fd}}{K_1} (1 - e^{-\frac{K_1}{T'_{do}}}) + e'_q e^{-\frac{K_1}{T'_{dqo}}} \]  \hspace{1cm} (5.243)

and

\[ e'_d = e'_d e^{-\frac{K_2}{T'_{dqo}}} \]  \hspace{1cm} (5.244)

where

\[ K_1 = \frac{x_d + x_e - x_c}{x'_d + x_e - x_c} \quad \text{and} \quad K_2 = \frac{x_q + x_e - x_c}{x'_q + x_e - x_c} \]

For stability, both \( K_1 \) and \( K_2 \) must be positive. Self excitation will occur when the system parameters are such that \( K_1 \) and/or \( K_2 \) become negative. Conditions for self excitation are therefore

\[ x_d + x_e \geq x_e \geq x'_d + x_c \]  \hspace{1cm} (5.245)

\[ x_q + x_e \geq x_e \geq x'_q + x_c \]  \hspace{1cm} (5.246)

It can be seen that as the capacitance is gradually increased starting with a very low value, self excitation will first occur in the \( d \) axis, since the condition for self excitation will be satisfied first in that axis.
Automatic voltage regulator action effectively reduces the direct axis synchronous reactance. Therefore, in the presence of automatic voltage regulators self excitation may start in the $q$ axis, since the condition for self excitation may be satisfied first in that axis.

In the above analysis, variation of machine speed was neglected. Self excitation is usually accompanied by considerable increase in machine speed. If the deviation from normal speed is excessive it should be included in the analysis, since it will have a significant impact on the onset of self excitation. It can be shown that at off-nominal speed the conditions for self excitation are approximately given by

$$\frac{\omega}{\omega_o} (L_d + L_c) \geq \frac{\omega}{\omega_o} C \geq \frac{\omega}{\omega_o} (L'_d + L_c)$$ (5.247)

for the direct axis, and

$$\frac{\omega}{\omega_o} (L_q + L_c) \geq \frac{\omega}{\omega_o} C \geq \frac{\omega}{\omega_o} (L'_q + L_c)$$ (5.248)

for the quadrature axis,

where $\omega$ is the actual speed and $\omega_o$ is the nominal speed.

**Problem**

1. Derive the conditions for self excitation including the effect of machine speed variation.

2. For the system shown in Figure 5.12 investigate the possibility of self excitation following opening of the breaker at the receiving end. If condition for self excitation is found to exist, estimate the size of the shunt reactor, to be placed at the high voltage side of the step-up transformer, that will avoid it. Assume a 5% machine over-speed.

![Fig. 5.12 A generator connected to a large system through a long distance transmission line.](image)

Another manifestation of the self excitation phenomenon caused by transients at subharmonic frequencies that may arise due to the application of series capacitors is discussed in Chapter 9. The analysis presented there is quite general and includes the self excitation discussed in this section.

**Derivation of Equations (5.144) - (5.146)**

Rewriting the direct axis flux linkage and rotor voltage equations from (5.60) - (5.62), considering one damper winding on each axis,

$$\psi_d = -x_{\alpha d} i_d + x_{\alpha dl} i_{ld} + x_{\alpha id} i_{id}$$ (5.249)

$$\psi_{ld} = -x_{\alpha id} i_d + x_{\alpha ld} i_{ld} + x_{f id} i_{id}$$ (5.250)

$$\psi_{ld} = -x_{\alpha id} i_d + x_{f id} i_{ld} + x_{11d} i_{id}$$ (5.251)
SYNCHRONOUS MACHINES

\[ e_{fd} = \frac{1}{\omega_o} \frac{d}{dt} \psi_{fd} + r_{fd} i_{fd} \]  
(5.252)

\[ 0 = \frac{1}{\omega_o} \frac{d}{dt} \psi_{1d} + r_{1d} i_{1d} \]  
(5.253)

Substituting \( \psi_{fd} \) and \( \psi_{1d} \) from (5.250) and (5.251) into (5.252) and (5.253) respectively, then eliminating \( i_{fd} \) and \( i_{1d} \) from the resulting equations, and using the Laplace operator \( s \) for \( \frac{d}{dt} \), the following is obtained.

\[ \psi_d = \frac{1 + T_{kd}s}{1 + (T_1 + T'_{do})s + T_d' T_d s^2} E_{fd} - \frac{1}{1 + (T_1 + T'_{do})s + T_d' T_d s^2} \]  
(5.254)

where \( T_{do}, T'_{do}, T_d', T_d'' \) and \( E_{fd} \) are as defined earlier, and

\[ T_{kd} = \frac{x_{1d}}{\omega_o r_{1d}}, \quad T_1 = \frac{x_{11d}}{\omega_o r_{1d}}, \quad T_2 = \frac{x_{11d} - x_{ad}^2}{\omega_o r_{1d}} \]

Note that (5.254) can also be approximated as

\[ \psi_d = \frac{1 + T_{kd}s}{(1 + T_1' s)(1 + T_d'' s)} E_{fd} - \frac{(1 + T_d' s)(1 + T_d'' s)}{(1 + T_1' s)(1 + T_d'' s)} x_d i_d \]  
(5.255)

(The expression for \( \psi_d \) in this form will be used in Chapter 9.)

Similarly, for the quadrature axis

\[ \psi_q = -\frac{1 + T_2'' s}{1 + T_2'' s} x_q i_q \]  
(5.256)

Neglecting the \( \psi \) terms and speed deviations, the armature voltage equations are

\[ e_d = -\psi_q - r i_d \]  
(5.257)

\[ e_q = \psi_d - r i_q \]  
(5.258)

From (5.256) and (5.117)

\[ T_2'' s (\psi_q + x_q i_q) = -(\psi_q + x_q i_q) \]

Setting

\[ \psi_q + x_q i_q = -e_d'' \]  
(5.259)

\[ T_2'' s e_d'' = (x_q - x_q^*) i_q - e_d'' \]  
(5.146)

Similarly, from (5.254), and using (5.114) and (5.115),

\[ T_{do}' T_{do} s^2 (\psi_d + x_d i_d) + s \left[ (T_1 + T_{do}' \psi_d + (T_2 + T_d') x_d i_d - T_{kd} E_{fd} \right] = E_{fd} - (\psi_d + x_d i_d) \]
Setting

\[ \psi_d + x_d^n i_d = e_q^n \]  

(5.260)

the above can be written as

\[ T_{do}' T_{do}' s^2 e_q^n + s \left[ (T_1 + T_{do}') (e_q^n - x_d^n i_d) + (T_2 + T_d') x_d i_d - T_{kd} E_{jd} \right] = E_{fd} - (x_d - x_d^n) i_d - e_q^n \]

Using an auxiliary variable

\[ e_q' = T_{do}' s e_q^n + \frac{1}{T_{do}'} \left[ (T_1 + T_{do}') (e_q^n - x_d^n i_d) + (T_2 + T_d') x_d i_d - T_{kd} E_{jd} \right] \]

it follows that

\[ T_{do}' s e_q' = E_{fd} - (x_d - x_d^n) i_d - e_q^n \]  

(5.261)

and, using (5.115),

\[ T_{do}' s e_q^n = e_q' - (x_d - x_d^n) i_d - e_q^n - \frac{1}{T_{do}'} \left[ T_1 (e_q^n - x_d^n i_d) + T_2 x_d i_d - T_{kd} E_{jd} \right] \]

(5.262)

Equations (5.261) and (5.262) can be written in a slightly modified form after neglecting the less significant fourth term on the right hand side of equation (5.262) and approximately compensating its effect by changing \( e_q^n \) to \( e_q' \) and \( x_d^n \) to \( x_d' \) in (5.261) to obtain equations (5.144) and (5.145). Numerous tests have shown this to be a valid approximation.

From (5.257) – (5.260)

\[ e_d^n = e_d + r i_d - x_d^n i_d \]  

(5.263)

\[ e_q^n = e_q + r i_q + x_d^n i_d \]  

(5.264)

References

CHAPTER 6
EFFECT OF EXCITATION CONTROL ON STABILITY

When a power system is subjected to a large disturbance such as a fault, the terminal voltages of the generators are depressed significantly and the generators' ability to transfer electrical power is reduced. Fast acting excitation controls can aid in system stability by quickly boosting the generators' field, thereby holding the terminal voltages to as high a level as possible. During the brief period immediately following a fault an excitation system with high speed and high ceiling voltage will have the most beneficial effect on the system. Following the clearing of the fault, if the machines have survived the first impact of the disturbance, they will go into oscillations. Complex interactions among the various machines and excitation and other control equipment can, under certain conditions, cause the oscillations to grow resulting in system instability. The requirement on the performance of the excitation control during this period is quite different from that during the initial period, and these requirements are often conflicting.

Under small-disturbance conditions, such as a load impact, the situation is similar to that existing beyond the initial period following a large disturbance, after the system has withstood the initial shock of the disturbance and has entered into a state of oscillation. For satisfactory performance it is necessary that the system and control parameters are such that sufficient damping is provided.

Since for small-disturbance studies we are concerned with small excursions of system variables about their steady state values, linearization of system equations is permissible and we can study the resulting linear system using the methods of linear system analysis.

Effect of Excitation on Generator Power Limit

In order to illustrate the potential of the excitation system in extending stability limit, let us consider the system shown in Figure 1, where a generator is supplying power to a large system represented by an equivalent machine of zero impedance and infinite inertia. The machine and system parameter values are as shown in the figure.

![Fig. 6.1 A one machine-infinite bus system.](image)

The power output of the machine is given by

\[ P = \frac{E_1 E_2}{X_1 + X_2} \sin \delta = \frac{E_1 E_2}{2} \sin \delta \]  

(6.1)

where

\[ \delta = \delta_1 - \delta_2 \]

The phasor diagram at a given load at unity terminal voltage and power factor is shown in Figure 6.2. At this operating condition \( E_1 \) and \( E_2 \) are equal as shown on the phasor diagram. If \( E_1 \) and \( E_2 \)
are held constant at these values, the maximum power transfer occurs when $\delta = 90^\circ$ as seen from equation (6.1).

Now assume perfect excitation control on both machines such that the terminal voltage, $V_t$, and the power factor are maintained at unity. This will be done by adjusting $E_1$ and $E_2$ instantaneously following any change in the load level.

From the phasor diagram of Figure 6.2, $\delta_1 = \delta_2 = \delta/2$, and therefore

$$E_1 = E_2 = \frac{V_t}{\cos \delta/2} = \frac{1}{\cos \delta/2}$$  \hspace{1cm} (6.2)

Substituting (6.2) into (6.1), we obtain

$$P = \frac{1}{2 \cos^2 \delta/2} \sin \delta = \tan \delta/2$$  \hspace{1cm} (6.3)

From equation (6.3) we can see that the power transfer can increase without limit. Of course, the excitation control as assumed above is not realizable in practice, since there is always a time lag in the excitation response. Also, the excitation control of the infinite bus voltage is not practical. There is also a maximum limit to which a machine's internal voltage can be raised.

Now consider a more practical situation where the infinite bus voltage, $E_2$, is held constant and the power factor is allowed to vary. Assuming both the terminal voltage, $V_t$, and the infinite bus voltage, $E_2$, held constant at 1.0 pu, we can draw the phasor diagram as shown in Figure 6.3.

Fig. 6.3 Phasor diagram for unity terminal and infinite bus voltage.

From the phasor diagram, since $V_t$ and $E_2$ are held constant and equal to 1.0 pu,
EFFECT OF EXCITATION ON STABILITY

\[ I / 2 = V_1 \sin(\delta / 2) \]

or

\[ I = 2 \sin(\delta / 2) \] (6.4)

\( E_1 \) can be obtained by adding the \( IX_1 \) drop to the terminal voltage \( V_t \). Taking \( V_1 \) as reference

\[ E_1 = 1 + jX_1[I \cos(\delta / 2) - jI \sin(\delta / 2)] = 1 + I \sin(\delta / 2) + jI \cos(\delta / 2) \] (6.5)

Since \( V_t \) and \( E_2 \) are held constant, it can be seen that the maximum power transfer occurs, from the relationship \( P = \frac{E_1 V_t}{X_2} \sin \delta \), when \( \delta = 90^\circ \).

Therefore, at maximum power transfer,

\[ E_1 = 1 + I / \sqrt{2} + jI / \sqrt{2} \]

From (6.4)

\[ I = 2 / \sqrt{2} = \sqrt{2} \]

Therefore

\[ E_1 = 2 + j = 2.235 \angle 26.6^\circ \]

and

\[ \delta = 90^\circ + 26.6^\circ = 116.6^\circ \]

Thus, in this case there is a limit to the power that can be transmitted and the maximum power transfer occurs at a value of \( \delta \) well over 90°. Again, in this illustration instantaneous control of the excitation voltage, \( E_1 \), has been assumed.

Effect of Excitation Control on Small-Disturbance Performance of Synchronous Machines

We will now analyze the effect of excitation control on the small-disturbance performance of a synchronous machine connected to an infinite bus through an external reactance. We will study the effects of exciter gain and time constants on machine stability. A more rigorous synchronous machine model than used in the previous elementary discussion will be employed. The synchronous machine model used is based on the two-axis representation as discussed in Chapter 5. The effects of damper windings will first be neglected. The natural damping caused by the various damper windings will be evaluated later on.

Neglecting damper windings, the equations for the synchronous machine (model 3) are as follows:

\[ T_{do} \frac{de_q'}{dt} = E_{fd} - (x_d' - x_d) i_d' - e_q' \] (6.6)

\[ \frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = P_m - \frac{K_d}{\omega_o} \frac{d\delta}{dt} - P_e \] (6.7)

\[ P_e = e_q' i_q' + (x_q' - x_q) i_d i_q \]

\[ e_d = x_q i_q \] (6.9)
EFFECT OF EXCITATION ON STABILITY

\[ e_q = e_q' - x_d' i_d \]  
(6.10)

We also have

\[ e_d = -x_d i_q + V_b \sin \delta \]  
(6.11)

\[ e_q = x_e i_d + V_b \cos \delta \]  
(6.12)

The phasor diagram of the synchronous machine-infinite bus system is shown in Figure 6.4.

![Phasor diagram of synchronous machine-infinite bus system](image)

Fig. 6.4 Phasor diagram of a synchronous machine connected to infinite bus through a reactance.

The terminal voltage is given by

\[ V_t^2 = e_d^2 + e_q^2 \]  
(6.13)

From (6.10) and (6.12), we have

\[ i_d = \frac{e_q'}{x_e + x_d'} - \frac{V_b \cos \delta}{x_e + x_d'} \]  
(6.14)

Similarly, from (6.9) and (6.11), we have

\[ i_q = \frac{V_b \sin \delta}{x_e + x_q} \]  
(6.15)

For small-disturbance analysis the equations can be linearized about the operating point. Linearizing equations (9.6) through (6.10) and (6.13) through (6.15), we have

\[ T_{do} \frac{d\Delta e'_d}{dt} = \Delta E_{fsl} - (x_d - x_d') \Delta i_d - \Delta e'_q \]  
(6.16)

\[ 2H \frac{d^2 \Delta \delta}{dt^2} = \frac{K_d}{\omega_o} \frac{d\Delta \delta}{dt} - \Delta P_c \]  
(6.17)

\[ \Delta P_c = \Delta e'_q i_q + e'_q \Delta i_q + (x_q - x_q')[\Delta i_d i_q + i_d \Delta i_q] \]  
(6.18)

\[ \Delta e_d = x_q \Delta i_q \]  
(6.19)

\[ \Delta e_q = x_q' \Delta i_d \]  
(6.20)

\[ V_t \Delta V_t = e_d \Delta e_d + e_q \Delta e_q \]  
(6.21)
EFFECT OF EXCITATION ON STABILITY

\[ \Delta i_d = \frac{1}{x_e + x_d} \Delta e_q' + \frac{V_b \sin \delta}{x_e + x_d} \Delta \delta \]  
(6.22)

\[ \Delta i_q = \frac{V_b \cos \delta}{x_v + x_q} \Delta \delta \]  
(6.23)

\[ i_d, i_q, e_d, e_q, \text{ etc.} \] are the values at the operating point and the prefix \( \Delta \) denotes the small changes in the variables about the operating point.

Substituting (6.22) and (6.23) into (6.18), we have

\[ \Delta P_e = \Delta e_q' i_q + \left[ e_q' + (x_v - x_d) i_d \right] \frac{V_b \cos \delta}{x_v + x_q} \Delta \delta 
+ (x_v - x_d) i_q \left[ \frac{1}{x_v + x_d} \Delta e_q' + \frac{V_b \sin \delta}{x_v + x_d} \Delta \delta \right] \]

which can be written as

\[ \Delta P_e = K_1 \Delta \delta + K_2 \Delta e_q' \]  
(6.24)

where

\[ K_1 = \left[ e_q' + (x_v - x_d) i_d \right] \frac{V_b \cos \delta}{x_v + x_q} + \frac{x_v - x_d}{x_v + x_d} i_q \frac{V_b \sin \delta}{x_v + x_d} \]  
(6.25)

\[ K_2 = \frac{x_v + x_q}{x_v + x_d} i_q \]  
(6.26)

Substituting (6.22) into (6.16) and using the Laplace variable \( s \) for \( d/dt \), we have

\[ T_{do} s \Delta e_q' = \Delta E_{fd} - \frac{1}{K_3} \Delta e_q' - K_4 \Delta \delta \]

where

\[ K_3 = \frac{x_v + x_d}{x_v + x_d} \]  
(6.27)

and

\[ K_4 = \frac{x_d - x_d'}{x_v + x_d'} V_b \sin \delta \]  
(6.28)

The above equation can be rearranged as

\[ \Delta e_q' = \frac{K_3}{1 + K_3 T_{do} s} \Delta E_{fd} - \frac{K_3 K_4}{1 + K_3 T_{do} s} \Delta \delta \]  
(6.29)

Substituting (6.22) and (6.23) into (6.20) and (6.19), respectively, and then the results in (6.21), we obtain

\[ \Delta V_t = K_5 \Delta \delta + K_6 \Delta e_q' \]  
(6.30)

where
EFFECT OF EXCITATION ON STABILITY

\[ K_5 = \frac{x_q e_d}{x_c + x_q} V_b \cos \delta - \frac{x_q'}{x_c + x_q'} \frac{e_q}{V_t} V_b \sin \delta \]  \hspace{1cm} (6.31)

and

\[ K_6 = \frac{x_c e_q}{x_c + x_q'} V_t \]  \hspace{1cm} (6.32)

It may be seen that all the constants \( K_1 \) through \( K_6 \), except \( K_3 \), are functions of the machine operating point. Since, as we will see, the stability characteristics are dependent on these constants, the system performance can be quite different at different operating points for the same exciter settings. Note that \( K_5 \) can become negative under heavily loaded condition (i.e., for high values of \( \delta \)).

The initial values of the various variables at an operating point may be calculated from the given operating condition as follows.

Referring to Figure 6.4, let the power output \( P \), the terminal voltage \( V_t \), and the infinite bus voltage \( V_b \) be given. Then

\[ P = \frac{V_t V_b}{x_c} \sin \alpha \]  \hspace{1cm} (6.33)

\[ Q = \frac{V_t^2 - V_i V_b \cos \alpha}{x_c} \]  \hspace{1cm} (6.34)

Therefore

\[ \alpha = \sin^{-1} \frac{P x_c}{V_b V_t} \]  \hspace{1cm} (6.35)

and

\[ \phi = \tan^{-1} \frac{Q}{P} \]  \hspace{1cm} (6.36)

Also

\[ P - jQ = V_i \hat{I}, \text{ with } V_i \text{ as reference} \]

\[ \therefore \hat{I} = \frac{P}{V_t} - j \frac{Q}{V_t} \]

\[ \hat{E}_q = V_i + j x_q \left( \frac{P}{V_i} - j \frac{Q}{V_i} \right) \]

\[ \therefore \theta = \tan^{-1} \frac{x_q P}{V_i^2 + x_q Q} \]  \hspace{1cm} (6.37)

\[ \delta = \theta + \alpha \]  \hspace{1cm} (6.38)

Also

\[ I = \frac{P}{V_t \cos \phi} \]  \hspace{1cm} (6.39)

\[ \therefore i_d = I \sin(\theta + \phi) \]  \hspace{1cm} (6.40)
The linearized model of the synchronous machine-infinite bus system described by equations (6.17), (6.24), (6.29) and (6.30) is suitable for estimating machine damping under various operating conditions. By adding to this model the equations representing the various controls acting on the machine, we can evaluate the effects of the controls on machine damping.

A schematic representation of the excitation control system is shown in Figure 6.5, where \( G(s) \) represents the combined transfer function of the regulator and the exciter.

![Fig. 6.5 Schematic of the excitation control.](image)

From Figure 6.5 the linearized equation of the excitation control system can be written as

\[-G(s) \Delta V_r = \Delta E_{jd}\] (6.45)

**Synchronizing power coefficient with constant flux linkage**

For constant flux linkage \( \Delta e'_q = 0 \). Therefore, from (6.24), \( \Delta P_e = K_1 \Delta \delta \). The synchronizing power coefficient for constant flux linkage is therefore

\[\frac{\Delta P_e}{\Delta \delta} = K_1\] (6.46)

From (6.17), the natural frequency of oscillation for constant flux linkage is given by

\[\omega_n = \sqrt{\omega_0 \frac{K_1}{2H}}\] (6.47)

**Steady-state synchronizing power coefficient**

(i) **No voltage regulator action or constant field voltage**

For constant field voltage, \( \Delta E_{jd} = 0 \). From equation (6.29), in the steady-state,

\[\Delta e'_q = -K_3 K_4 \Delta \delta\]

Therefore, from equation (6.24),

\[\Delta P_e = (K_1 - K_3 K_5 K_4) \Delta \delta\]

from which the synchronizing power coefficient is
(ii) **Effect of saturation**

In order to keep the analysis simple, certain valid simplifying assumptions will be made. These are:

1. Saturation on the $d$ axis can be expressed as a function of $e'_q \left[ S = f(e'_q) \right]$ only.
2. $x_q$ (saturated value) is not affected appreciably by small changes in the operating point.

Therefore, the equations for the synchronous machine will be identical to equations (6.6) - (6.12), except for equation (6.6) which will be replaced by

$$T_{do} \frac{de'_q}{dt} = E_{fd} - (x_d - x'_d) i_d - (1 + S) e'_q$$

(6.49)

where

$$S = f(e'_q) \quad (\text{see Chapter 5})$$

Note that, in these equations, all the reactances, except $x_q$, have unsaturated values.

Proceeding as before, after linearization of equations (6.7) - (6.12) and some algebraic manipulations, we obtain

$$\Delta P_e = K_1 \Delta \delta + K_2 \Delta e'_q$$

(6.24)

Linearizing equation (6.49), we have

$$T_{do} \frac{d\Delta e'_q}{dt} = \Delta E_{fd} - (x_d - x'_d) \Delta i_d - (1 + k) \Delta e'_q$$

(6.50)

where

$$k = S + \frac{\partial S}{\partial e'_q} e'_q$$

In the steady-state, assuming constant field voltage, $\Delta E_{fd} = 0$,

$$(x_d - x'_d) \Delta i_d + (1 + k) \Delta e'_q = 0$$

Substituting the value of $\Delta i_d$ from (6.22) and simplifying,

$$\Delta e'_q = - \frac{K_3 K_4}{1 + k K_3} \Delta \delta$$

(6.51)

From (6.24) and (6.51), the steady-state synchronizing power coefficient can be derived as

$$\frac{\Delta P_e}{\Delta \delta} = K_1 - \frac{K_2 K_3 K_4}{1 + k K_3}$$

(6.52)

Comparing (6.52) with (6.48), it can be seen that the effect of saturation is to increase the synchronizing power coefficient and, hence, extend the stability limit.
(iii) **With voltage regulator action**

From (6.29), in the steady-state

\[ \Delta e'_{q} = K_3 \Delta E_{fd} - K_5 K_4 \Delta \delta \]  

(6.53)

Assuming that the excitation control is represented by a simple gain and time lag

\[ G(s) = \frac{K_e}{1 + \tau_e s} \]

then, from equation (6.30) and (6.45),

\[ \Delta E_{fd} = -K_e \left( K_5 \Delta \delta + K_6 \Delta e_{q}' \right) \]  

(6.54)

since in the steady-state \( \Delta V_{ref} = 0 \)

Substituting (6.54) into (6.53) and simplifying,

\[ \Delta e_{q}' = -\frac{K_1 K_4 + K_3 K_2 K_e}{1 + K_3 K_2 K_e} \Delta \delta \]  

(6.55)

Substituting (6.55) into (6.24), we have

\[ \frac{\Delta P_e}{\Delta \delta} = K_1 - K_2 \frac{K_3 K_2 K_e}{1 + K_3 K_2 K_e} \]  

(6.56)

Comparing (6.56) with (6.46), it is seen that for a value of \( K_e \) equal to \(-K_4/K_2\) the synchronizing power coefficient will be equal to that for constant flux linkage.

**Evaluation of damping torque**

(i) **No regulator action**

With no voltage regulator action \( \Delta E_{fd} = 0 \), and therefore from equation (6.29)

\[ \Delta e_{q}' = -\frac{K_2 K_3 K_4}{1 + K_3 T_{do} s} \Delta \delta \]  

(6.57)

Substituting (6.57) into (6.24), we have

\[ \Delta P_e = \left[ K_1 - \frac{K_2 K_3 K_4}{1 + K_3 T_{do} s} \right] \Delta \delta \]  

(6.58)

For a frequency of oscillation \( \omega, s \) can be replaced by \( j\omega \); then (6.58) reduces to

\[ \Delta P_e = \left[ K_1 - \frac{K_2 K_3 K_4}{1 + j\omega K_3 T_{do}} \right] \Delta \delta \]  

(6.59)

The imaginary part of the above equation represents the damping torque component (the component in quadrature with angle or in phase with speed). Note that at normal speed, the per unit power and per unit torque are equal. Therefore, the damping torque is given by

\[ \text{Damping torque} = T_d j \Delta \delta \]  

(6.60)
where
\[ T_d = \frac{\omega K_2 K_3^2 K_4 T_{do}'}{1 + \omega^2 K_3^2 T_{do}'} \] (6.61)

The damping torque is also expressed as
\[ \text{Damping torque} = D \frac{d\Delta \delta}{dt} = D s \Delta \delta \]

where \( D \) is the equivalent damping constant.

At oscillation frequency \( \omega \), replacing \( s \) by \( j \omega \), the above can be written as
\[ \text{Damping torque} = D \omega j \Delta \delta \] (6.62)

Comparing (6.62) with (6.60), we can write
\[ D = T_d / \omega \] (6.63)

The per unit damping coefficient is, therefore
\[ K_d = D \omega_o = (T_d / \omega) \omega_o \] (6.64)

It may be seen from equation (6.59) that the synchronizing power coefficient is reduced by

\[ \frac{K_2 K_3 K_4}{1 + \omega^2 K_3^2 T_{do}'} \]

due to the demagnetizing effect of armature reaction. From (6.59), (6.60) and (6.61) it may be concluded that, since \( K_2, K_3 \) and \( K_4 \) are normally positive, a certain amount of positive damping is introduced by the demagnetizing effect of armature reaction.

In situations where \( \omega^2 K_3^2 T_{do}'^2 >> 1 \), (6.61) reduces to
\[ T_d = \frac{K_2 K_4}{\omega T_{do}'} \]

and therefore
\[ K_d = \frac{\omega_o K_2 K_4}{\omega^2 T_{do}'} \] (6.65)

Substituting the expressions for \( K_2 \), \( K_3 \) and \( K_4 \) in (6.61), the expression for the damping coefficient can also be written as
\[ K_d = \frac{\omega_o V_p^2 (x_d - x_d') T_{do}'}{(x_e + x_d')^2 + \omega^2 (x_e + x_d')^2 T_{do}'^2} \] (6.66)

From (6.59), the synchronizing power coefficient is given by
\[ \frac{\Delta P_e}{\Delta \delta} = K = K_1 = \frac{K_2 K_3 K_4}{1 + \omega^2 K_3^2 T_{do}'} \] (6.67)

Therefore, from (6.17), the natural frequency of oscillation in the absence of voltage regulator action is given by
Example

As an example, consider the system shown in Figure 4 with the following parameter values, representative of a typical nuclear generator.

\[ x_d = 1.48, \quad x_q = 1.42, \quad x'_d = 0.35, \quad T'_{do} = 6.0, \quad H = 4.0, \quad x_e = 0.3 \]

All values, except \( T'_{do} \), are in per unit on machine rated MVA base. Assume an operating point where \( P = 1.0, \quad V_t = 1.06 \) and \( V_b = 1.05 \).

Initial operating point values are, from equations (6.35) through (6.46),

\[
\begin{align*}
\omega_0 &= 9.61, \\
\omega_1 &= 0.13, \\
\omega_2 &= 3.35, \\
\omega_3 &= 7.0, \\
\omega_4 &= 7.631, \\
\omega_5 &= 5.375, \\
\omega_6 &= 7.923, \\
\delta &= 61.69^- \quad \text{rad/s} \\
K_1, K_2, K_3, K_4, K_5, K_6 &= 1.3566, 1.4223, 0.3652, 1.6071, -0.0495, 0.3203 \\
K_d &= 2.32 \\
K_d' &= 2.33 \\
\end{align*}
\]

The natural frequency calculated from equation (6.68) is approximately 7.9 rad/sec. The per unit damping coefficient computed from equations (6.61) and (6.64) is \( K_d = 2.32 \). The damping coefficient computed from the approximate expression, (6.65), is \( K_d' = 2.33 \).

It may be seen, from (6.61) and (6.64), that at very low frequency of oscillations \( (\omega \approx 0) \), the damping coefficient becomes

\[ K_d \approx \omega_0 K_2 K_3^2 K_4 T'_{do} \]

(ii) Including voltage regulator action

Substituting (6.30) into (6.45), and the result in (6.29), with \( \Delta V_{\text{ref}} = 0 \), we obtain

\[
\Delta e'_q = \frac{-K_3}{1 + K_3 T'_{do} s} [K_5 \Delta \delta + K_6 \Delta e'_q] G(s) - \frac{K_3 K_4}{1 + K_3 T'_{do} s} \Delta \delta
\]

which yields

\[ \Delta e'_q = -\frac{K_3 K_4}{1 + K_3 T'_{do} s} \frac{1 + K_2 G(s)}{1 + K_3 K_6 G(s)} \Delta \delta \]

A comparison of the above expression with equation (6.57) shows that, under operating conditions where \( K_5 \) is negative, the action of excitation control with transfer functions commonly encountered in power systems will result in a reduction in the machine damping produced by demagnetizing effect of armature reaction.

Following the steps outlined earlier, the damping torque coefficient is obtained from (6.24) and (6.70) as
\[
K_d = \frac{\omega_o}{\omega} \text{Im} \left[ -K_2K_4 \frac{1 + \frac{K_5}{K_4}G(j\omega)}{1 + j\omega K_5 T_{do}'} \right]
\] (6.71)

For an exciter control with a simple gain and time lag, the transfer function \( G(s) \) is represented by \( G(s) = \frac{K_e}{1 + \tau_e s} \), and the damping torque constant is obtained as

\[
T_d = \frac{\omega \left( K_2K_5 K_e + K_2K_4 \right)(T_{do}' + \tau_e / K_3) - \omega K_2K_4 \tau_e \left( \frac{1}{K_3} + K_6 K_e - \omega^2 T_{do}' \tau_e \right)}{\left( \frac{1}{K_3} + K_6 K_e - \omega^2 T_{do}' \tau_e \right)^2 + \omega^2 \left( T_{do}' + \tau_e / K_3 \right)^2}
\] (6.72)

**Example**

In the previous example, consider a fast acting high gain excitation control system having an equivalent transfer function given by

\[
G(s) = \frac{50}{1 + 0.1s}
\]

which yields

\[ G(j\omega) = 39.33 - 38.13^\circ \]

Performing the numerical computations, we have

\[-\frac{K_3K_4}{1 + j\omega K_3 T_{do}'} = -0.034 - 86.67^\circ \]

and

\[ \frac{1 + \frac{K_5}{K_4}G(j\omega)}{1 + j\omega K_5 T_{do}'} = 0.856 - 100.83^\circ \]

This example clearly demonstrates the mechanism of the reduction of machine damping by conventional excitation control. It can be seen that, in this particular case, the resultant damping is negative. Completing the computation, we have

\[ K_d = \frac{\omega_o}{\omega} \text{Im}[ -K_2 \times 0.034 - 86.67^\circ \times 0.856 - 100.83^\circ ] = -0.48 \]

Recall that for fixed excitation the damping coefficient would be

\[ K_d = \frac{\omega_o}{\omega} \text{Im}[ -K_2 \times 0.034 - 86.67^\circ ] = 2.23 \]

Next, consider a slow acting low gain excitation control system having an equivalent transfer function given by
EFFECT OF EXCITATION ON STABILITY

\[ G(s) = \frac{10}{1 + s} \quad \text{or} \quad G(j\omega) = 1.264 \angle -82.74^\circ \]

Performing the necessary computations, we have

\[ K_d = \frac{\omega_0}{\omega} \text{Im}[ -K_2 \times 0.034 \angle -86.67^\circ \times 1.\angle 2.3^\circ ] = 2.31 \]

Thus, the reduction in damping due to this particular slow acting excitation control is negligible.

The transfer functions of actual excitation control systems are much more complex than the simple transfer function used in the above example. However, at a particular frequency of oscillation \( \omega \), all transfer functions are easily reducible to the form \( G(j\omega) = |G|\angle \theta \) and, therefore, can be readily incorporated in the numerical computation.

As an example, consider the IEEE type 1 excitation system shown in block diagram form in Figure 6.6a

Fig. 6.6a Block diagram of the IEEE type 1 excitation system.

In linearized form this reduces to the block diagram shown in Figure 6.6b

Fig. 6.6b Linearized block diagram of the system shown in Fig. 6.6a.
EFFECT OF EXCITATION ON STABILITY

Applying the block diagram reduction technique (see Appendix B), the above can be reduced to the form

![Diagram](image)

Fig. 6.7 Reduced form of the block diagram shown in Fig. 6.6.

where

\[
G(s) = \frac{K_A(1 + T_F s)}{K_E}\frac{K_A s + (1 + T_A s)(K'_E + T_E s)(1 + T_F s)}
\]

\[
K'_E = K_E + (1 + BE_{fd}) S_E
\]

Example

Consider the following set of parameter values.

\[
K_A = 400, \quad T_A = 0.01, \quad K_E = 1.0, \quad T_E = 1.079, \quad K_F = 0.079, \quad T_F = 1.0
\]

\[
S_E \text{ at } 75\%E_{fd \text{ max}} = 0.568, \quad S_E \text{ at } E_{fd \text{ max}} = 1.322
\]

\[
E_{fd \text{ max}} = 4.122
\]

Using these parameter values and setting \(s = j\omega = j7.85\), \(G(j\omega)\) is computed as

\[
G(j\omega) = 11.5^\circ - 21.2^\circ
\]

Using this value of \(G(j\omega)\), the damping coefficient in the previous example is computed as

\[
K_d = \frac{\omega}{\omega} \text{Im}[-K_2 \times 0.034 \angle -86.67^\circ \times 0.697 \angle 15.17^\circ] = 1.54
\]

Note the reduction in damping due to the action of the excitation control.

Next, consider the following values

\[
K_A = 69.5, \quad T_A = 0.01, \quad K_E = 1.0, \quad T_E = 3.2, \quad K_F = 0.143, \quad T_F = 1.0
\]

\[
S_E \text{ at } 75\%E_{fd \text{ max}} = 0.087, \quad S_E \text{ at } E_{fd \text{ max}} = 0.875
\]

\[
E_{fd \text{ max}} = 5.991
\]

For this set of parameter values, \(G(j\omega)\) is computed as

\[
G(j\omega) = 2.5\angle -71.5^\circ
\]

The damping coefficient is computed from

\[
K_d = \frac{\omega}{\omega} \text{Im}[-K_2 \times 0.034 \angle -86.67^\circ \times 0.995 \angle 4.7^\circ] = 2.3
\]

Note that the reduction in damping is negligible.
Effects of Damper Windings

(a) One damper winding on the q axis, no damper winding on the d axis

The equations for the synchronous machine with one damper winding on the q axis (model 2) are:

\[ T'_{do} \frac{de'_q}{dt} = E_{jd} - (x_d - x'_d) i_d - e'_q \]  
(6.73)

\[ T'_{qo} \frac{de'_d}{dt} = (x_q - x'_q) i_q - e'_d \]  
(6.74)

\[ P_e = e'_d i_d + e'_q i_q + (x'_q - x'_d) i_d i_q \]  
(6.75)

\[ e_d = e'_d + x'_d i_q \]  
(6.76)

\[ e_q = e'_q - x'_d i_d \]  
(6.77)

From (6.76), (6.77) and (6.11), (6.12), we obtain

\[ i_d = \frac{-e'_q}{x_c + x'_d} - \frac{V_b \cos \delta}{x_c + x'_d} \]  
(6.78)

\[ i_q = \frac{e'_d}{x_c + x'_q} + \frac{V_b \sin \delta}{x_c + x'_q} \]  
(6.79)

Linearization of equations (6.73) through (6.79) yields

\[ T'_{do} \frac{d\Delta e'_q}{dt} = \Delta E_{jd} - (x_d - x'_d) \Delta i_d - \Delta e'_q \]  
(6.80)

\[ T'_{qo} \frac{d\Delta e'_d}{dt} = (x_q - x'_q) \Delta i_q - \Delta e'_d \]  
(6.81)

\[ \Delta P_e = \Delta e'_d i_d + e'_d i_d + \Delta e'_q i_q + e'_q \Delta i_q + (x'_q - x'_d) (\Delta i_d i_q + i_d \Delta i_q) \]  
(6.82)

\[ \Delta e_d = \Delta e'_d + x'_q \Delta i_q \]  
(6.83)

\[ \Delta e_q = \Delta e'_q - x'_d \Delta i_d \]  
(6.84)

\[ \Delta i_d = \frac{1}{x_c + x'_d} \Delta e'_q + \frac{V_b \sin \delta}{x_c + x'_d} \Delta \delta \]  
(6.85)

\[ \Delta i_q = -\frac{1}{x_c + x'_q} \Delta e'_d + \frac{V_b \cos \delta}{x_c + x'_q} \Delta \delta \]  
(6.86)

Substituting (6.85) and (6.86) into (6.82) and rearranging, we obtain

\[ \Delta P_e = K_1 \Delta \delta + K_2 \Delta e'_q + K_3 \Delta e'_d \]  
(6.87)

where
EFFECT OF EXCITATION ON STABILITY

\[ K_1 = \left[ e_d' + (x_q' - x_d') i_q \right] \frac{V_b \sin \delta}{x_e + x_d'} + \left[ e_q' + (x_q' - x_d') i_d \right] \frac{V_b \cos \delta}{x_e + x_q'} \]  
\[ \text{(6.88)} \]

\[ K_2 = \frac{e_d'}{x_e + x_d'} + \frac{x_e + x_d'}{x_e + x_d'} i_q = \frac{V_b \sin \delta}{x_e + x_d'} \]  
\[ \text{(6.89)} \]

\[ K_9 = -\frac{e_q'}{x_e + x_q'} + \frac{x_e + x_q'}{x_e + x_q'} i_q = -\frac{V_b \cos \delta}{x_e + x_q'} \]  
\[ \text{(6.90)} \]

Substituting (6.85) into (6.80) and (6.86) into (6.81), and rearranging, we obtain

\[ \Delta e_d' = \frac{K_3}{1 + K_3 T_{do}' s} \Delta E_{jd} - \frac{K_3 K_4}{1 + K_3 T_{do}' s} \Delta \delta \]  
\[ \text{(6.91)} \]

where

\[ K_3 = \frac{x_e + x_d'}{x_e + x_d} \]  
\[ \text{(6.92)} \]

\[ K_4 = \frac{x_q - x_d'}{x_e + x_d'} V_b \sin \delta \]  
\[ \text{(6.93)} \]

and

\[ \Delta e_d' = \frac{K_4 K_8}{1 + K_7 T_{qo}' s} \Delta \delta \]  
\[ \text{(6.94)} \]

where

\[ K_7 = \frac{x_e + x_q'}{x_e + x_q} \]  
\[ \text{(6.95)} \]

\[ K_8 = \frac{x_q - x_q'}{x_e + x_q'} V_b \cos \delta \]  
\[ \text{(6.96)} \]

Substituting (6.85) and (6.86) into (6.84) and (6.83), respectively, and the results in (6.21), and rearranging, we obtain

\[ \Delta V_i = K_5 \Delta \delta + K_6 \Delta e_q' + K_{10} \Delta e_d' \]  
\[ \text{(6.97)} \]

where

\[ K_5 = \frac{x_q' e_d}{x_e + x_q'} V_b \cos \delta - \frac{x_d' e_q}{x_e + x_d'} V_b \sin \delta \]  
\[ \text{(6.98)} \]

\[ K_6 = \frac{x_q e_q}{x_e + x_q'} V_t \]  
\[ \text{(6.99)} \]

\[ K_{10} = \frac{x_e e_d}{x_e + x_q'} V_t \]  
\[ \text{(6.100)} \]
Evaluation of damping torque

(i) No voltage regulator action

With no voltage regulator action, $\Delta E_{fd} = 0$. Therefore, substituting (6.91) and (6.94) into (6.87), we obtain

$$
\Delta P_e = \left[ K_1 - \frac{K_2 K_3 K_4}{1 + K_3 T_{do} s} + \frac{K_7 K_s K_8}{1 + K_7 T_{qo} s} \right] \Delta \delta
$$

(6.101)

The third term in the above expression is due to the $q$ axis damper winding. $K_7$ is positive and the product $K_8 K_9$ is negative. Therefore, the effect of the $q$ axis damper winding is to provide additional damping.

From equation (6.101), for an oscillatory frequency $\omega$, the damping coefficient is obtained as:

$$
K_d = \frac{\omega_o K_2 K_3^2 K_4 T'_{do}}{1 + \omega^2 K_3^2 T_{do}^2} \frac{\omega_o K_7 K_8 K_9 T'_{qo}}{1 + \omega^2 K_7^2 T_{qo}^2}
$$

(6.102)

The first term in the above expression is due to the field winding and the second term is due to the $q$ axis damper winding.

Substituting the expressions for $K_2, K_3$ etc. in (6.102), the damping coefficient can also be expressed as

$$
K_d = \frac{\omega_o V_b^2 (x_d - x'_d) T'_{do} \sin^2 \delta}{(x_e + x_d)^2 + \omega^2 (x_e + x'_d)^2 T_{do}^2} + \frac{\omega_o V_b^2 (x_q - x'_q) T'_{qo} \cos^2 \delta}{(x_e + x_q)^2 + \omega^2 (x_e + x'_q)^2 T_{qo}^2}
$$

(6.103)

Example

Consider the system used in the previous example, with the additional data due to the $q$ axis damper winding given as

$$
x'_q = 0.5, \quad T'_{qo} = 0.5
$$

$K_1$ through $K_{10}$ are computed from equations (6.88) through (6.100), and are obtained as:

$$
K_1 = 1.5223, \quad K_2 = 1.4222, \quad K_3 = 0.3652, \quad K_4 = 1.6071, \quad K_5 = -0.1214
$$

$$
K_6 = 0.3203, \quad K_7 = 0.4651, \quad K_8 = 0.5726, \quad K_9 = -0.6224, \quad K_{10} = 0.27
$$

Using these values, the two terms of the right-hand side of equation (6.102) are computed as:

$$
\frac{\omega_o V_b^2 K_2 K_3^2 K_4 T'_{do}}{1 + \omega^2 K_3^2 T_{do}^2} = 2.323
$$

$$
- \frac{\omega_o V_b^2 K_7 K_8 K_9 T'_{qo}}{1 + \omega^2 K_7^2 T_{qo}^2} = 3.354
$$

Therefore, the damping coefficient, obtained by adding the two terms, is: $K_d = 5.677$

Note that the contribution to damping due to the $q$ axis damper winding is substantial.
(ii) **Effect of voltage regulator**

The linearized equation of the excitation control system is given by equation (6.45). Substituting (6.97) into (6.45), and the result in (6.91), we obtain

\[
\Delta e'_q = \frac{-K_3}{1 + K_3 T_{do} s} \left[ K_5 \Delta \delta + K_6 \Delta e'_q + K_{10} \Delta e'_d \right] G(s) - \frac{K_3 K_4}{1 + K_3 T_{do} s} \Delta \delta
\]

which yields, using (6.94),

\[
\Delta e'_q = \frac{-K_3 K_4}{1 + K_3 T_{do} s} \left[ 1 + \frac{1}{K_4} \left( K_5 + \frac{K_7 K_8 K_{10}}{1 + K_7 T_{do} s} \right) G(s) \right] \Delta \delta
\]

Comparing this expression with that given in (6.70), it is seen that the effect of the \( q \) axis damper winding is to add an extra term in the numerator. As before, damping coefficient is obtained from (6.87), (6.104) and (6.94), as

\[
K_d = \frac{\omega_0}{\omega} \text{Im} \left[ \frac{-K_3 K_4}{1 + j \omega K_3 T_{do}} \left( 1 + \frac{1}{K_4} \left( K_5 + \frac{K_7 K_8 K_{10}}{1 + j \omega K_7 T_{do} s} \right) G(j \omega) \right) + \frac{K_3 K_4 K_9}{1 + j \omega K_3 T_{do} s} G(j \omega) \right]
\]

(6.105)

Note that the excitation control action does not affect the damping on the \( q \) axis, although the presence of the \( q \) axis damper winding influences the change in damping on the \( d \) axis due to excitation control action.

**Example**

Consider the excitation control parameters as used in a previous example. \( G(j \omega) \) was computed as:

\[ G(j \omega) = 11.5 \angle -21.2 \]

Performing the computation as indicated in (6.105), the \( d \) axis component of the damping coefficient is

\[ K^d_d = 0.5 \]

Note that the reduction in damping is substantial

The \( q \) axis component of the damping coefficient is not affected by excitation control and, as computed previously, is

\[ K^q_d = 3.354 \]

Therefore, the total damping is

\[ K_d = 3.854 \]

Next, consider the excitation control parameters where \( G(j \omega) \) is

\[ G(j \omega) = 2.5 \angle -71.5 \]
Performing the computation in (6.105),
\[ K_d^d = 2.11 \]
The total damping is therefore
\[ K_d = 2.11 + 3.354 = 5.464 \]
The reduction in damping in this case is negligible.

(b) **One damper winding on the \( d \) axis, two damper windings on the \( q \) axis**

The equations of the synchronous machine with one damper winding on the \( d \) axis and two damper windings on the \( q \) axis are (here we use the equations in the form of equations (5.144) – (5.146) of Chapter 5, with one additional damper winding added to the \( q \) axis.)

\[
T'_{do} \frac{de'_q}{dt} = E_{fd} - (x_d - x'_d) i_d - e'_q \quad (6.106)
\]
\[
T''_{do} \frac{de''_q}{dt} = e'_q - (x'_d - x''_d) i_d - e''_q \quad (6.107)
\]
\[
T'_{dq} \frac{de'_d}{dt} = (x_q - x'_q) i_d - e'_d \quad (6.108)
\]
\[
T''_{dq} \frac{de''_d}{dt} = e'_q + (x'_q - x''_q) i_q - e''_d \quad (5.109)
\]
\[
P_e = e'_d i_d + e''_q i_q + (x''_q - x'_q) i_d i_q \quad (6.110)
\]
\[
e_d = e'_d + x'_q i_q = e''_q + x''_q i_q \quad (6.111)
\]
\[
e_q = e'_q - x'_q i_d = e''_q - x''_q i_d \quad (6.112)
\]

From (6.111), (6.112) and (6.11), (6.12) we obtain
\[
i_d = \frac{e'_q}{x_c + x'_d} - \frac{V_b \cos \delta}{x_c + x'_d} = \frac{e''_q}{x_c + x''_d} - \frac{V_b \cos \delta}{x_c + x''_d} \quad (6.113)
\]
\[
i_q = \frac{-e'_d}{x_c + x'_q} + \frac{V_b \sin \delta}{x_c + x'_q} = \frac{-e''_d}{x_c + x''_q} + \frac{V_b \sin \delta}{x_c + x''_q} \quad (6.114)
\]

Linearization of (6.106) through (6.114) yields
\[
T'_{do} \frac{d\Delta e'_q}{dt} = \Delta E_{fd} - (x_d - x'_d) \Delta i_d - \Delta e'_q \quad (6.115)
\]
\[
T''_{do} \frac{d\Delta e''_q}{dt} = \Delta e'_q - (x'_d - x''_d) \Delta i_d - \Delta e''_q \quad (6.116)
\]
\[
T'_{dq} \frac{d\Delta e'_d}{dt} = (x_q - x'_q) \Delta i_d - \Delta e'_d \quad (6.117)
\]
\[
T''_{dq} \frac{d\Delta e''_d}{dt} = \Delta e'_d + (x'_q - x''_q) \Delta i_d - \Delta e''_d \quad (6.118)
\]
\[
\Delta P_e = \Delta e''_q i_d + e''_q \Delta i_d + \Delta e''_q i_q + e''_q \Delta i_q + (x''_q - x'_q)(\Delta i_d i_q + i_d \Delta i_q) \quad (6.119)
\]
\[ \Delta e_d = \Delta e'_d + x'_q \Delta i_q = \Delta e''_d + x''_q \Delta i_q \quad (6.120) \]

\[ \Delta e_q = \Delta e'_q - x'_d \Delta i_d = \Delta e''_d - x''_d \Delta i_d \quad (6.121) \]

\[ \Delta i_d = \frac{1}{x_e + x'_q} \Delta e'_d + \frac{V_b \sin \delta}{x_e + x'_q} \Delta \delta = \frac{1}{x_e + x'_q} \Delta e''_d + \frac{V_b \sin \delta}{x_e + x'_q} \Delta \delta \quad (6.122) \]

\[ \Delta i_q = -\frac{1}{x_e + x'_q} \Delta e'_d + \frac{V_b \cos \delta}{x_e + x'_q} \Delta \delta = -\frac{1}{x_e + x'_q} \Delta e''_d + \frac{V_b \cos \delta}{x_e + x'_q} \Delta \delta \quad (6.123) \]

Substituting (6.122) and (6.123) into (6.119), and rearranging, we obtain

\[ \Delta P_e = K_{15} \Delta \delta + K_{16} \Delta e''_d + K_{17} \Delta e'_d \quad (6.124) \]

where

\[ K_{15} = \left[ e''_d + (x''_q - x''_d) i_q \right] \frac{V_b \sin \delta}{x_e + x'_q} + \left[ e''_q + (x''_q - x''_d) i_d \right] \frac{V_b \cos \delta}{x_e + x'_q} \quad (6.125) \]

\[ K_{16} = \frac{e''_d}{x_e + x'_q} + \frac{x_e + x''_d}{x_e + x'_q} i_q = \frac{V_b \sin \delta}{x_e + x'_q} \quad (6.126) \]

\[ K_{17} = -\frac{e''_q}{x_e + x'_q} + \frac{x_e + x''_d}{x_e + x'_q} i_d = -\frac{V_b \cos \delta}{x_e + x'_q} \quad (6.127) \]

Substituting (6.122) first in (6.115) and then in (6.116), and rearranging, we obtain

\[ \Delta e'_d = \frac{K_3}{1 + K_3 T'_{do}s} \Delta E_{fd} - \frac{K_3 K_4}{1 + K_3 T'_{do}s} \Delta \delta \quad (6.128) \]

where \( K_3 \) and \( K_4 \) are as given by (6.92) and (6.93), respectively, and

\[ \Delta e''_q = \frac{K_{11}}{1 + K_{11} T''_{do}s} \Delta e'_d - \frac{K_{11} K_{12}}{1 + K_{11} T''_{do}s} \Delta \delta \quad (6.129) \]

where

\[ K_{11} = \frac{x_e + x''_d}{x_e + x'_d} \quad (6.130) \]

\[ K_{12} = \frac{x'_d - x''_d}{x_e + x'_d} V_b \sin \delta \quad (6.131) \]

Substitution of (6.128) into (6.129) yields

\[ \Delta e''_q = \frac{K_3 K_{11}}{(1 + K_3 T'_{do}s)(1 + K_{11} T''_{do}s)} \Delta E_{fd} - \frac{K_3 K_4 K_{11}}{(1 + K_3 T'_{do}s)(1 + K_{11} T''_{do}s)} \Delta \delta \quad (6.132) \]

In a similar way, from (6.117), (6.118) and (6.123), we obtain

\[ \Delta e''_d = \left[ \frac{K_7 K_9 K_{13}}{(1 + K_7 T'_{qo}s)(1 + K_{13} T''_{qo}s)} + \frac{K_{13} K_{14}}{1 + K_{13} T''_{qo}s} \right] \Delta \delta \quad (6.133) \]
where $K_7$ and $K_8$ are given by (6.95) and (6.96)

$$K_{13} = \frac{x_e + x'_q}{x_e + x''_q}$$  \hspace{1cm} (6.134)

$$K_{14} = \frac{x'_q - x''_q}{x_e + x''_q} V_b \cos \delta$$  \hspace{1cm} (6.135)

**Evaluation of damping torque**

(i) *No voltage regulator action*

With no voltage regulator action, $\Delta E_{fd} = 0$. Therefore, substituting (6.132) and (6.133) into (6.124), we obtain

$$\Delta \Phi = \begin{bmatrix} K_{15} - \frac{K_3 K_4 K_{11} K_{16}}{(1 + K_3 T''_{do}) s} (1 + K_{11} T''_{do} s) - \frac{K_{11} K_{12} K_{16}}{1 + K_{11} T''_{do} s} \\
+ \frac{K_7 K_8 K_{13} K_{17}}{(1 + K_7 T''_{qo} s) (1 + K_{13} T''_{qo} s) s} + \frac{K_{13} K_{14} K_{17}}{1 + K_{13} T''_{qo} s} s \end{bmatrix} \Delta \delta$$  \hspace{1cm} (6.136)

Since $T''_{do} \ll T'_{do}$ and $T''_{qo} \ll T'_{qo}$, the above expression can be approximated as

$$\Delta \Phi = \begin{bmatrix} K_{15} - \frac{K_3 K_4 K_{11} K_{16}}{(1 + K_3 T'_{do} s) s} (1 + K_{11} T'_{do} s) - \frac{K_{11} K_{12} K_{16}}{1 + K_{11} T'_{do} s} \\
+ \frac{K_7 K_8 K_{13} K_{17}}{(1 + K_7 T'_{qo} s) (1 + K_{13} T'_{qo} s) s} + \frac{K_{13} K_{14} K_{17}}{1 + K_{13} T'_{qo} s} s \end{bmatrix} \Delta \delta$$  \hspace{1cm} (6.137)

Substituting the expressions for $K_3$, $K_4$, etc. in the above expression, the damping torque coefficient at oscillation frequency $\omega$ is obtained as

$$K_d = \omega q V_b \left[ \frac{(x'_d - x'_d') T'_{do}}{x_e + x'_d + \omega^2 (x_e + x'_d)^2 T_{do}^2} + \frac{(x'_q - x''_q) T''_{qo}}{(x_e + x'_q + \omega^2 (x_e + x'_q)^2 T_{qo}^2} \right] \sin^2 \delta$$

$$+ \omega q V_b \left[ \frac{(x'_q - x''_q) T''_{qo}}{x_e + x'_q + \omega^2 (x_e + x'_q)^2 T_{qo}^2} + \frac{(x'_q - x''_q) T''_{qo}}{(x_e + x'_q + \omega^2 (x_e + x'_q)^2 T_{qo}^2} \right] \cos^2 \delta$$  \hspace{1cm} (6.138)

The first term is due to the field winding and the $d$ axis damper winding, and the second term is due to the $q$ axis damper windings.

**Example**

Consider the previous example with the following additional parameter values:

$$x'_d = 0.18, \quad x''_d = 0.2, \quad T''_{fd} = 0.025, \quad T''_{qo} = 0.05$$

Substituting these and the other parameter values in (6.138), we obtain the $d$ axis component of the damping coefficient
and the $q$ axis component
\[ K_q^d = 5.42 \]
Therefore the total damping is
\[ K_d = K_d^d + K_q^d = 10.92 \]
Note that the contribution to damping due to the additional damper winding is substantial.

(ii) **Effect of voltage regulator**

The effect of voltage regulator can be included in the analysis following the procedure described earlier. However, in the presence of damper windings on both $d$ and $q$ axis, the expression for the damping coefficient becomes extremely cumbersome. It would be more expedient to work out the individual steps and arrive at the final value of the damping coefficient numerically. As before, it may be noted that the voltage regulator has no effect on the $q$ axis component of machine damping, although the $q$ axis damper windings will contribute to the reduction of the $d$ axis component of damping due to voltage regulator action. As before, it can be shown that the presence of the $q$ axis damper winding will result in a greater reduction in the $d$ axis component of damping due to voltage regulator action.

**Supplementary Stabilizing Signals**

As seen earlier, certain excitation and system parameter combinations under certain loading conditions can introduce negative damping into the system. In order to offset this effect and to improve system damping in general, artificial means of producing torques in phase with the speed are introduced. These are called supplementary stabilizing signals and the networks used to generate these signals are known as power system stabilizers. For example, a fast acting, high-gain voltage regulator, although useful for improving transient stability margin, often depletes the generators' natural damping, thus rendering the system response highly oscillatory. When the use of such a high-response regulator-exciter system is indicated from a transient stability consideration, the resulting system oscillations can be minimized or eliminated by employing power system stabilizers.

Stabilizing signals are introduced at the point where the reference voltage and the signal proportional to the terminal voltage are compared to obtain the error signal. The signal, usually obtained from speed, frequency or accelerating power, is processed through a suitable network to obtain the desired phase relationship. Such an arrangement is shown schematically in Figure 6.8.

Fig. 6.8 A schematic diagram of a stabilizing signal from speed deviation
A typical power system stabilizer using speed signal is shown in Figure 6.9. A transducer converts the signal to a voltage. The signal, depending on where in the shaft system it is measured and picked up by the transducer, may contain, in addition to the generator swing mode, a number of torsional modes (for a discussion on generator shaft torsionals see Chapter 9). The high frequency filters are designed to attenuate these torsional frequencies as well as high frequency measurement noise. The lead-lag stages (usually identical) are used to provide an overall phase lead over the frequency range of interest to compensate for the lag produced in the generator-excitation system. Most PSS applications use two stages of phase compensation. In some applications three stages have been employed to obtain additional phase lead, or one stage when less phase lead is required. The signal is then amplified and sent through a washout stage which prevents voltage offsets during steady-state or prolonged speed or frequency change. The output limiter prevents the stabilizer output signal from interfering with the action of the voltage regulator during severe system disturbances.

\[
\frac{\Delta \omega}{\omega_o} = \left( \frac{1 + A_1 s + A_2 s^2}{1 + A_3 s + A_4 s^2} \right) \left( \frac{1 + A_5 s + A_6 s^2}{1 + A_7 s + A_8 s^2} \right) \]

\( \text{High Frequency Filters} \)

\[
\frac{1}{1 + s \tau_3} \quad \frac{1}{1 + s \tau_2} \quad \frac{K_S}{1 + s \tau_S} \quad \frac{V_S}{V_{S\text{MIN}}} \]

Fig. 6.9 A typical power system stabilizer

From Figure 6.8, the linearized equation of the exciter system, including the stabilizer, can be written as, for \( \Delta V_{ref} = 0 \)

\[
\begin{bmatrix} F(s) \frac{\Delta \omega}{\omega_o} - \Delta V_t \end{bmatrix} G(s) = \Delta E_{fd}
\]

where \( F(s) \) is the transfer function of the stabilizer.

Here we will use synchronous machine model 3 for the purpose of illustration. Substituting (6.139) into (6.29) and using (6.30), we obtain

\[
\Delta e_q' = \frac{K_3}{1 + K_3 T_{do} s} \left[ F(s) \frac{\Delta \omega}{\omega_o} - K_5 \Delta \delta - K_6 \Delta e_q' \right] G(s) - \frac{K_3 K_4}{1 + K_3 T_{do} s} \Delta \delta
\]

Obtaining the expression for \( \Delta e_q' \) from (6.140) and substituting into (6.24), we obtain the component of torque due to the stabilizing signal as

\[
\Delta P_e' = \frac{K_2 K_3 G(s)}{(1 + K_3 T_{do} s) \left[ 1 + \frac{K_4 K_3 G(s)}{1 + K_3 T_{do} s} \right]} F(s) \frac{\Delta \omega}{\omega_o}
\]

(6.141)
EFFECT OF EXCITATION ON STABILITY

The above can also be written as

$$\Delta P^* = G'(s) F(s) \frac{\Delta \omega}{\omega_o}$$

(6.142)

where

$$G'(s) = \frac{K_2 K_5 G(s)}{(1 + K_3 T_{do}s) \left[ 1 + \frac{K_3 K_5 G(s)}{1 + K_3 T_{do}'s} \right]}$$

(6.143)

At oscillation frequency $\omega$, equation (6.142) becomes

$$\Delta P^* = G'(j \omega) F(j \omega) \frac{\Delta \omega}{\omega_o}$$

(6.144)

Therefore, the damping component due to the stabilizer action is given by

$$K'_d = \text{Re}\left[G'(j \omega) F(j \omega)\right]$$

(6.145)

which can also be written as

$$K'_d = \text{Re}\left[G' \angle \theta' |F| \angle \theta_s\right]$$

(6.146)

Therefore, for maximum damping effect due to the stabilizer, the stabilizer parameters should be adjusted so as to have $\theta_s = -\theta'$. Since $\theta'$ is negative, the stabilizer network should ideally advance the signal by $-\theta'$. It should be noted that a selection of parameter values which provide insufficient lead or introduce a lag in the stabilizer can either render the stabilizer ineffective or contribute negative damping. The stabilizer must be tuned to provide the desired system performance under the conditions which require stabilization, typically weak systems under heavy power transfer. When there is only one frequency that exhibits poor or negative damping, it is a relatively simple matter to select the appropriate phase lead. However, in an interconnected system there may be multiple frequencies, that change with system and loading conditions, that need to be stabilized. The actual phase lead selected would therefore be a compromise to produce satisfactory performance at all these frequencies.

The amount of damping that can be provided by a PSS can be seen to be directly proportional to the gain. However, the gain cannot be selected arbitrarily. Above a certain value of the gain control loop instability sets in. (Note that, in the analysis above, only the rotor angle mode was considered in an idealized single machine-infinite bus system.) The measurement noise and torsional frequencies that are not completely filtered out can also produce undesirable effects unless the gain is limited. There is also the question of interactions among PSSs in a large interconnected system. The effect of the PSS on all modes of oscillation as well as the interactions among PSSs can be studied using a detailed system model as discussed in Chapter 8. It is generally recommended that the final selection of phase lead and gain be done in a field test. It is customary to select a gain about one-third of the value at which small rapid oscillation is just sustained.

Note that for normal values of $\omega$, $G'(j \omega)$, from (6.143), can approximately be written as

$$G'(j \omega) \approx \frac{K_2}{\omega T_{do}'} |G| \angle \theta - 90^\circ$$

(6.147)
since \[ \frac{K_x K_\omega G(j\omega)}{1 + j\omega K_3 T_{do}} \approx 1 \]

Therefore, a quick estimate of the required phase advance by the stabilizer network can easily be made.

It can also be seen from equation (6.147) that, due to the large combined lag in the machine and the regulator-exciter, it would be more convenient to employ accelerating power instead of speed as the stabilizing signal. This would then necessitate only a moderate amount of lead in the stabilizer network. Also, the power signal is generally free from shaft torsional modes.

References
CHAPTER 7
REPRESENTATION OF LOADS IN STABILITY STUDIES

Loads, for the purpose of stability analyses, represent the aggregate of innumerable number of individual component devices such as motors, lighting, and electrical appliances, unless they are of appreciable sizes requiring explicit representations. The aggregated load is usually the load as seen from bulk power delivery points, comprising several to tens of megawatts. In addition, the aggregated load model approximates the effects of sub-transmission and distribution system lines, cables, reactive power compensation, LTC transformer, distribution voltage regulators, and even relatively small synchronous or induction generators.

Load Characteristics
Load characteristics can be divided into two broad categories -- static and dynamic.

Static load: The active and reactive power drawn by the load at any instant of time are functions of bus voltage magnitude and frequency at the same instant. Examples are heating and lighting loads.

Dynamic load: The load whose responses to disturbances (changes of voltages and frequency) do not occur instantaneously, but require some time. They tend to recover to or close to their original level following changes in voltage and frequency within a certain range. Instantaneously, they behave as static loads; the recovery is governed by the overall time constants depending on the type of load. Prime examples are motor loads driving equipment with a variety of torque-speed characteristics.

Classes of Load
Loads can be classified as follows:

- loads that exhibit “fast dynamics” -- e.g., motor loads
- loads whose responses exhibit significant discontinuities -- e.g., discharge lighting, adjustable speed drives, motor contactors that open during low voltage, motor overload protection that removes stalled motors from the system after a time delay
- loads whose responses do not exhibit discontinuities -- e.g., very small motors, incandescent lamps, resistive loads
- loads with “slow dynamic” characteristics -- e.g., loads controlled thermostatically or manually

Basis Modeling Concepts
Accurate modeling of loads is a difficult task due to several factors, including:

- large number of diverse load components
- type and location of load devices in customer facilities not precisely known for system analysis purposes
- changing load composition with time of day, week, seasons, weather, and through time
lack of information on the composition of load

uncertainties regarding the characteristics of many load components, particularly for large voltage or frequency variations

In a qualitative manner, the relative behavior of bulk power system loads may be predicted from knowledge of the type of connected load. Agricultural and industrial loads are generally motor loads and should be expected to exhibit more dynamic characteristics than commercial and residential loads. Similarly, heavy summer loads resulting from air conditioning equipment will have more dynamic characteristics than winter heating and lighting loads.

**Static Model for Stability Studies**

In large-scale stability simulations loads are typically represented as functions of voltage and frequency. It is common to represent the active and reactive powers separately, by combination of constant impedance, constant current, and constant power elements -- the so-called “ZIP” representation. This may also be regarded as a polynomial representation, in which each respective element varies as the square of the voltage, the first power of the voltage, or not at all with voltage. The appropriate combination is often selected on the basis of matching the measured or estimated variation of load with voltage around nominal voltage. The variation with frequency is usually considered separately. The slopes of active and reactive power to voltage around nominal voltage are the best known and most generally available data, most of which are concerned with the voltage (and frequency) dependency for small variations in voltage around nominal values. The reactive power characteristics of many components are non-linear in voltage, and therefore, the slope values do not yield complete information. While the actual range of frequency dependency of interest is limited (5%) for stability studies, the range of voltage dependency of interest is much wider (0 - 120%). The static characteristics, which are based on a narrow range of voltage variations, are often used in studies involving wide range of voltage variations. It has been customary to convert the load to constant impedance when the voltage drops below some critical value, mainly for computational convenience.

Intuitively, the constant power load representation is the most severe (pessimistic) representation form the system stability point of view, because of its effect in amplifying voltage oscillations: a drop in voltage will cause an increase in load current resulting in a further voltage drop. Conversely, constant impedance loads would have a decided damping effect on voltage oscillations. However, the above is usually true in the cases where the loads are at major load centers remote from generation. The opposite can be true in the cases where loads are at the sending end of the transmission line.

Voltage dependency of reactive power affects stability primarily due to its effects on voltage, which in turn affect real power.

The frequency dependency is recognized as an important contributor to system damping. It is intuitively obvious that the more the active power of load decreases with decreasing frequency (i.e., a positive slope of power to frequency), the more stable the system. As for reactive power characteristics, a negative slope is best. This is because an increase of reactive load with decreasing frequency tends to depress the voltage and thus generally reduce the active power further. Both shunt reactors and shunt capacitors have this negative slope.

In some computer programs loads are represented as exponential functions of voltage, usually in the following form:
\[ P = P_0 \left( \frac{V}{V_0} \right)^{n_p}, \quad Q = Q_0 \left( \frac{V}{V_0} \right)^{n_q} \]

where \( P_0, Q_0, V_0 \) are the values at the initial system operating point.

Two or more terms with different exponents are sometimes included in each equation. The parameters in this model are the exponents, \( n_p \) and \( n_q \), and the power factor of the load. The exponents can be chosen to represent the aggregate effect of different types of load compositions. Exponent values 0, 1, or 2 correspond to constant power, constant current, or constant impedance loads.

Reference 1 recommends the following static model for dynamic simulations. This consists of ZIP terms plus two voltage/frequency dependent terms

\[ \frac{P}{P_{frac}P_0} = K_{pc} \left( \frac{V}{V_0} \right)^2 + K_{pi} \frac{V}{V_0} + K_{pc} + K_{p1} \left( \frac{V}{V_0} \right)^{n_{p1}} \left( 1 + n_{pf1} \Delta f \right) + K_{p2} \left( \frac{V}{V_0} \right)^{n_{p2}} \left( 1 + n_{pf2} \Delta f \right) \quad (7.1) \]

\[ K_{pc} = 1 - \left( K_{pi} + K_{pc} + K_{p1} + K_{p2} \right) \quad (7.2) \]

where \( P_{frac} \) is the fraction of the bus load represented by the static model.

\[ \frac{Q}{Q_{frac}Q_0} = K_{qc} \left( \frac{V}{V_0} \right)^2 + K_{qi} \frac{V}{V_0} + K_{qc} + K_{q1} \left( \frac{V}{V_0} \right)^{n_{q1}} \left( 1 + n_{qf1} \Delta f \right) + K_{q2} \left( \frac{V}{V_0} \right)^{n_{q2}} \left( 1 + n_{qf2} \Delta f \right) \quad (7.3) \]

\[ Q_0 \neq 0 \quad (7.4) \]

\[ K_{qc} = 1 - \left( K_{qi} + K_{qc} + K_{q1} + K_{q2} \right) \quad (7.5) \]

where \( Q_{frac} \) is the fraction of the bus load represented by the static model.

In these equations \( P_0 \) and \( Q_0 \) are the initial active and reactive load powers from power flow base case; they may be termed the nominal load powers, meaning the load power at initial voltage and frequency. \( P \) and \( Q \) are the consumed load powers as a function of voltage and frequency.

The bus frequency, \( f \), can be computed by taking the numerical derivative of the bus voltage angle. It is sometimes approximated by using the average system frequency, computed from a weighted average of synchronous machine speeds. This approximation is, however, not correct since it will not produce the correct impact on damping of oscillations.

**Discussion**

The parameters of the static models are either estimated or derived from tests that neglect dynamics. Failure to represent loads in sufficient detail may produce results that miss significant phenomena -- an example is voltage stability studies. For stability studies the load model should reflect the actual characteristics of the load under dynamic conditions.

In addition to realistically approximating the actual behavior of the load, the model must be physically realizable. For example, the constant \( P \) and constant \( I \) part in the ZIP model are not physically realizable, i.e., no real load behaves like these. These models, provided they reflect
the actual characteristics of the load under steady state, are good for steady state analyses only, although it is common practice to use constant \( I \) for active load and constant \( Z \) for reactive load in stability studies, and constant \( P \) for both active and reactive load for power flow studies. In the absence of accurate data on load characteristics, it is common to assume what is believed to be a pessimistic representation. However, a model that produces pessimistic results in some situations might produce just the opposite results in some other situations. Some examples are given in [2].

Some loads, especially the electronically controlled loads, recover to nominal voltage level quickly after a drop in voltage and so might appear to be proper candidates for constant \( P \) representation in stability studies. While in some situations this may not affect the study results, in others this might lead to misleading conclusions. This is not necessarily a problem in very low voltage situation since in most stability programs constant \( P \) load is changed to constant \( Z \) load below a threshold level for computational expediency.

In the ZIP model commonly used in dynamic analyses, the \( Z \) part is valid if the load that it represents is constant \( Z \). However, the \( I \) and \( P \) parts are not valid representations for any load in dynamic analyses. The reason is quite simple, although easily overlooked. No real load behaves as constant \( I \) or \( P \) in the dynamic state. Following a disturbance when there is a transient change in voltage, loads, that are constant \( I \) or \( P \) in the steady state, will also go through a transient change before recovering (if they can recover) to their steady-state level. The recovery may be fast or slow, but this needs to be accounted for in the model. Since the constant \( I \) and \( P \) models do not account for this transient, they are incompatible with the rest of the system model. The results obtained with the \( IP \) model are therefore theoretically invalid. They may look normal, and occasionally they may even be comparable to what would be obtained using a legitimate model for these loads. Usually, when the \( IP \) part is a small fraction of the total load, the discrepancy is negligible, but the problem is there.

In general dynamic simulation studies, due to the nonlinearities, it's difficult to show analytically the problem one can experience when using a theoretically invalid load model. However, let us perform this thought experiment. Consider a single generator transmitting power over a double (or triple) circuit transmission line (assumed loss-less) and serving a constant power load. (This one-machine-constant-power-load system has been used in voltage stability studies on many occasions and reported in the literature.) Since the line is loss-less, the generator output power is identically equal to the power drawn by the load. Now suppose a disturbance is created by tripping one of the lines. Since the load is constant power, the generator output also remains constant following the disturbance. The generator, therefore, does not feel the impact of the disturbance. The problem with constant power load should now be apparent. For small disturbance analyses, when the system is linearized, an analytical demonstration of the problem is possible [6]. Similarly, it can be shown that in the exponential load model an exponent value of 1.0 or less is invalid [7].

Normally the \( IP \) part of the ZIP model is used to represent the induction motor loads (and other loads that tend to recover to constant \( I \) or \( P \)) that are too numerous and whose characteristics and parameters are not readily available. This is neither necessary nor desirable. If the load is induction motor, it should be represented as such. If a bus is serving too many of them, they can be lumped, using whatever information is available. In the absence of accurate data, default model and parameter values can be used. At least it will be closer to reality. When the load is a large industrial motor load, the relevant data for any of the standard motor models would be available. As regards other constant (steady state) \( P \) and \( I \) loads, it may be possible to construct
suitable models based on their physical properties. Failing that, the simplified first-order induction motor model (discussed in a later section) can serve as a generic model for such loads. Although a large number of additional differential equations would be added to the system model, computational burden can actually be lower with such a model, since the iterations required in the network solutions due to the presence of \( IP \) loads would be eliminated. Most importantly, the results would be closer to reality.

The so-called "model validation" by comparing the simulation responses with those observed in actual disturbances may not mean much. It is not difficult to match the actual disturbance recordings, after the fact, by adjusting the load mix and parameter values (assuming the load model was causing the discrepancies). When the model is not physically correct, it cannot be expected to faithfully predict the impact of future disturbances.

**Motor Load**

The motor load usually is a large portion of the total load (more than 60% of the total electric energy is consumed by motors in the USA), and may have a large impact in system dynamics during certain disturbances. Synchronous motors can be represented by adopting the generator models with appropriate expressions for torque-speed characteristics. Representation of induction motor will be discussed in some detail in the next section. If the motor component of the load is represented as one or more motors, its behavior in response to voltage and frequency variations will be automatically obtained. However, it is not easy to find a single motor that will satisfactorily represent the aggregate motor load.

**Induction Motor**

In a typical power system up to 60% of the load served may be motor load, and of these the majority would be induction motors. Motors used in appliances are small, and in stability studies the usual practice is to aggregate the motors served by a sub-transmission or distribution bus and represent it either by an aggregate motor model or by a static model given by (7.1) - (7.5). Induction motors used in industrial applications are of large enough capacity to warrant detailed individual representation.

In this section we will develop induction motor model suitable for use in stability studies. Our approach will be similar to that adopted for synchronous machine modeling discussed in detail in Chapter 5.

In conventional induction motors the stator windings are connected to the source and the rotor windings are either short-circuited or closed through external resistances. Induction motors run below synchronous speed. The speed decreases with increasing load torque.

Since the induction motor has no inherent means for producing its excitation, it requires reactive power and therefore draws a lagging current. In order to limit the reactive power, the magnetizing reactance needs to be high, and the air gap is therefore shorter than in synchronous machines of the same size and rating.

The stator windings of polyphase induction motors are basically the same as the stator windings of polyphase synchronous machines. However, the induction motors fall into two general categories depending on the kind of rotor used -- the wound-rotor and the squirrel-cage rotor. The stator iron as well as the rotor iron is laminated and slotted to contain the windings. The wound-
rotor has a three-phase winding similar to that in the stator and is wound for the same number of poles as the stator winding. In the squirrel-cage rotor bars of copper or aluminum, known as rotor bars, occupy the slots. These are short-circuited in two end rings of the same material as the rotor bars. The rotating magnetic field produced by the three phase voltages applied to the stator windings induces currents in the squirrel-cage rotor circuit that develop the same number of rotor poles as stator.

For improved performance some squirrel-cage motors use deep-bar rotors or are equipped with two squirrel cages in the rotor.

For the purpose of analyses, both would-rotor and squirrel-cage rotor induction motors can be considered to have three stator and rotor windings shown schematically in Figure 7.1. Although more than one set of rotor windings may be needed to accurately account for the double-cage and deep-bar motors, for stability studies this is hardly necessary.

As in the case of synchronous machines, we can write the voltage equations as, assuming positive current for generator action,

\[ e_{s,abc} = \frac{d}{dt} \Psi_{s,abc} - r_s i_{s,abc} \quad (7.6) \]

where

\[ e_{s,abc} = \begin{bmatrix} e_{a1} \\ e_{b1} \\ e_{c1} \end{bmatrix}, \quad \Psi_{s,abc} = \begin{bmatrix} \psi_{a1} \\ \psi_{b1} \\ \psi_{c1} \end{bmatrix}, \quad i_{s,abc} = \begin{bmatrix} i_{a1} \\ i_{b1} \\ i_{c1} \end{bmatrix} \]

Similarly,

\[ e_{r,abc} = 0 = \frac{d}{dt} \Psi_{r,abc} + r_r i_{r,abc} \quad (7.7) \]

where
Inductance relationships can be obtained exactly as in the case of synchronous machines, noting that in the induction motor the air-gap is uniform, and so the inductances are not functions of rotor position.

\[
\begin{align*}
l_{aa1} &= l_{bb1} = l_{cc1} = l_s \\
l_{ab1} &= l_{bc1} = l_{ca1} = l_s \cos 120^\circ = -\frac{1}{2} l_s \\
l_{aa2} &= l_{bb2} = l_{cc2} = l_r \\
l_{ab2} &= l_{bc2} = l_{ca2} = -\frac{1}{2} l_r \\
l_{al1} &= l_{bl2} = l_{cl2} = l_m \cos \theta \\
l_{al2} &= l_{bl2} = l_{cl2} = l_m \cos(\theta - 120^\circ) \\
l_{al3} &= l_{bl2} = l_{cl2} = l_m \cos(\theta + 120^\circ) \\
\end{align*}
\]

The flux linkage relationship can therefore be written as

\[
\begin{align*}
\Psi_{s,abc} &= -L_{ss} i_{s,abc} + L_{sr} i_{r,abc} & (7.8) \\
\Psi_{r,abc} &= -L_{sr} i_{s,abc} + L_{rr} i_{r,abc} & (7.9)
\end{align*}
\]

where

\[
L_{ss} = l_s \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}, \quad L_{rr} = l_r \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}
\]

and

\[
L_{sr} = l_m \begin{bmatrix} \cos \theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\ \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) & \cos \theta \\ \cos(\theta + 120^\circ) & \cos \theta & \cos(\theta - 120^\circ) \end{bmatrix}
\]

We will use similar transformation as in the analysis of synchronous machine, by resolving the voltages, currents, and flux linkages along a set of \(d-q\) axes fixed on the rotor, the \(d\) axis coinciding with the axis of \(a_2\), and, as before, introducing an additional variable to make the transformation reversible. However, since there are two sets of windings, we will need two sets of transformations.

Using the symbols \(T_s\) and \(T_r\) for the transformation matrices for the stator and rotor, respectively,
\[ T_s = \frac{2}{3} \begin{bmatrix} \cos \theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\ -\sin \theta & -\sin(\theta - 120^\circ) & -\sin(\theta + 120^\circ) \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \]  

(7.10)

\[ T_s^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta & 1 \\ \cos(\theta - 120^\circ) & -\sin(\theta - 120^\circ) & 1 \\ \cos(\theta + 120^\circ) & -\sin(\theta + 120^\circ) & 1 \end{bmatrix} \]  

(7.11)

\[ \theta \text{ is function of time, since the } d-q \text{ axes are rotating at the speed of the rotor, and therefore} \]

\[ \theta = \omega t \]  

(7.12)

where \( \omega \) is the angular speed of the rotor.

Since the \( d-q \) axes are fixed on the rotor, the rotor windings are stationary with respect to the \( d-q \) axes. Therefore

\[ T_r = \frac{2}{3} \begin{bmatrix} 1 & \cos 120^\circ & \cos 120^\circ \\ 0 & -\sin 120^\circ & \sin 120^\circ \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \]  

(7.13)

\[ T_r^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1/2 & -\sqrt{3}/2 & 1 \\ -1/2 & \sqrt{3}/2 & 1 \end{bmatrix} \]  

(7.14)

Applying the transformation to the stator and rotor quantities,

\[ e_{s,dqo} = T_s e_{s,abc} \]

\[ i_{s,dqo} = T_s i_{s,abc} \]

\[ \Psi_{s,dqo} = T_s \Psi_{s,abc} \]

\[ e_{r,dqo} = T_r e_{r,abc} \]

\[ i_{r,dqo} = T_r i_{r,abc} \]

\[ \Psi_{r,dqo} = T_r \Psi_{r,abc} \]  

(7.15)

Using (7.15), (7.8) can be written as

\[ T_s^{-1} \Psi_{s,dqo} = -L_{ss} T_s^{-1} i_{s,dqo} + L_{sr} T_r^{-1} i_{r,dqo} \]

or

\[ \Psi_{s,dqo} = -T_s L_{ss} T_s^{-1} i_{s,dqo} + T_s L_{sr} T_r^{-1} i_{r,dqo} \]

After carrying out the indicated matrix operations, and noting that for balanced operation \( \psi_{so} = i_{so} = i_{ro} = 0 \), we obtain

\[ \psi_{sd} = -\frac{3}{2} l_s i_{sd} + \frac{3}{2} l_m i_{rd} \]  

(7.16)
\[
\Psi_{sq} = -\frac{3}{2} l_s i_{sq} + \frac{3}{2} l_m i_{rq} 
\]

or, in matrix form,
\[
\Psi_{s,dq} = -L_s i_{s,dq} + M i_{r,dq} 
\]

where
\[
L_s = \frac{3}{2} l_s, \quad M = \frac{3}{2} l_m
\]

Similarly, from (7.9) and (7.15) we obtain
\[
\Psi_{r,dq} = -M i_{s,dq} + L_r i_{r,dq} 
\]

where
\[
L_r = \frac{3}{2} l_r
\]

From (7.6), using (7.15),
\[
T_s^{-1} e_{s,dq} = \frac{d}{dt}[T_s^{-1}\Psi_{s,dq}] - r_s T_s^{-1} i_{s,dq}
\]

or
\[
e_{s,dq} = T_s \frac{d}{dt}[T_s^{-1}\Psi_{s,dq}] - r_s i_{s,dq}
\]

After carrying out the indicated matrix operation, we obtain, for balanced operation,
\[
e_{s,dq} = \frac{d}{dt}\Psi_{s,dq} + \omega W \Psi_{s,dq} - r_s i_{s,dq}
\]

where \( \omega = \dot{\theta} \) = rotor speed, and
\[
W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Similarly, from (7.7), and using (7.15),
\[
T_r^{-1} e_{r,dq} = \frac{d}{dt}[T_r^{-1}\Psi_{r,dq}] + r_r T_r^{-1} i_{r,dq}
\]

or
\[
e_{r,dq} = \frac{d}{dt}\Psi_{r,dq} + r_r i_{r,dq}
\]

from which, since \( e_{rd} = e_{rq} = 0 \),
\[
0 = \frac{d}{dt}\Psi_{r,dq} + r_r i_{r,dq}
\]

Power input is given by
\[
P = e'_{s,abc} i_{s,abc} = e'_{s,dqo} [T_s^{-1}]' T_s^{-1} i_{s,dq} = \frac{3}{2} (e_{sd} i_{sd} + e_{sq} i_{sq})
\]

since \( e_{so} = i_{yo} = 0 \)
Assume a balanced three phase voltages, \( e_a = E \cos(\omega_o t) \), \( e_b = E \cos(\omega_o t - 120^\circ) \), \( e_c = E \cos(\omega_o t + 120^\circ) \), applied at the machine terminals, producing currents, \( i_a = I \cos(\omega_o t - \phi) \), \( i_b = I \cos(\omega_o t - 120^\circ - \phi) \), \( i_c = I \cos(\omega_o t + 120^\circ - \phi) \). Applying the transformations (7.15), the \( d-q \) axes voltages and currents are obtained as

\[
e_d = E \cos(\omega_o - \omega) t \\
e_q = E \sin(\omega_o - \omega) t \\
i_d = I \cos((\omega_o - \omega) t - \phi) \\
i_q = I \sin((\omega_o - \omega) t - \phi)
\] (7.23, 7.24, 7.25, 7.26)

The \( d-q \) axes voltages and currents are thus alternating at the slip frequency \( \omega_o - \omega \), where \( \omega_o \) is the synchronous speed.

Equations (7.18) - (7.21) are referred to \( d-q \) axes rotating at the speed of the rotor, \( \omega \). Since the network equations are expressed in terms of the \( D-Q \) axes rotating at synchronous speed, \( \omega_o \), it is desirable to refer the induction machine equations to the same reference frame as the network equations. This is done by using a transformation similar to that used to refer synchronous machines to a common network reference frame. The transformation used is given by

\[
\begin{bmatrix}
D \\
Q
\end{bmatrix} = T \begin{bmatrix}
d \\
q
\end{bmatrix}
\] (7.27)

where

\[
T = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\] and \( T^{-1} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}\)

\( D-Q \) refer to the synchronously rotating network reference frame. Note that in the above transformations \( \theta \) is a function of time, since the \( D-Q \) axes are rotating at the speed \( (\omega_o - \omega) \) with respect to the rotor. Thus \( \theta = (\omega_o - \omega) t \), or

\[
\dot{\theta} = \omega_o - \omega
\] (7.28)

Applying the transformation to equations (7.18) through (7.21), in turn,

\[
T^{-1} \Psi_{s,DQ} = -L_s T^{-1} i_{s,DQ} + M T^{-1} i_{r,DQ}
\] or

\[
\Psi_{s,DQ} = -L_s i_{s,DQ} + M i_{r,DQ}
\] (7.29)

Similarly,

\[
\Psi_{r,DQ} = -M i_{s,DQ} + L_r i_{r,DQ}
\] (7.30)

From (7.20)

\[
T^{-1} e_{s,DQ} = \frac{d}{dt} (T^{-1} \Psi_{s,DQ}) + \omega W T^{-1} \Psi_{s,DQ} - r_s T^{-1} i_{s,DQ}
\] or
After carrying out the indicated matrix operations, we obtain
\[ e_{s,DQ} = \frac{d}{dt}(T^{-1}\Psi_{s,DQ}) + \omega W T^{-1}\Psi_{s,DQ} - r_i i_{s,DQ} \quad (7.31) \]

From (7.21)
\[ 0 = \frac{d}{dt}(T^{-1}\Psi_{r,DQ}) + r_i T^{-1}i_{r,DQ} \]
or
\[ 0 = T \frac{d}{dt}(T^{-1}\Psi_{r,DQ}) + r_i i_{r,DQ} \]

After carrying out the indicated matrix operation, we obtain
\[ 0 = \frac{d}{dt}\Psi_{r,DQ} + \dot{\theta} W \Psi_{r,DQ} + r_i i_{r,DQ} \quad (7.32) \]

The power input is
\[ P = \frac{3}{2} e'_{s,DQ} i_{s,DQ} = \frac{3}{2} e_{s,DQ}[T^{-1}]^T i_{s,DQ} = \frac{3}{2} (e_{sD}i_{sD} + e_{sQ}i_{sQ}) \quad (7.33) \]

We obtain the electrical torque, as in the case of synchronous machine, as
\[ T_e = \frac{3}{2} \frac{\omega_o}{\omega} (\Psi_{sQ} i_{sQ} - \Psi_{sD} i_{sD}) \quad (7.34) \]

Using the same per unit system as used for synchronous machine, we convert equations (7.29) - (7.34) to per unit form, to obtain
\[ \Psi_s = -L_s i_s + M i_r \quad (7.35) \]
\[ \Psi_r = -M i_s + L_i i_r \quad (7.36) \]
\[ e_s = \frac{1}{\omega_o} \frac{d}{dt} \Psi_s + W \Psi_s - r_s i_s \quad (7.37) \]
\[ 0 = \frac{1}{\omega_o} \frac{d}{dt} \Psi_r + s W \Psi_r + r_i i_r \quad (7.38) \]

where \( s \) is the per unit slip
\[ s = \frac{\dot{\theta}}{\omega_o} = \frac{\omega_o - \omega}{\omega_o} \quad (7.39) \]

Note that in the above equations we have dropped \( DQ \) from the subscript for clarity.
\[ P = e_{sD} i_{sD} + e_{sQ} i_{sQ} \quad (7.40) \]
\[ T_e = \Psi_{sD} i_{sQ} - \Psi_{sQ} i_{sD} \quad (7.41) \]
Induction Motor Representation in Stability Studies

We now derive the induction motor equations suitable for use in stability studies. The effect of the $\frac{d}{dt}\Psi_s$ terms is small and can be neglected.

From (7.36)

$$i_r = \frac{1}{L_r} \Psi_r + \frac{M}{L_r} i_s \tag{7.42}$$

Substituting (7.42) in (7.35),

$$\Psi_s = \frac{M}{L_r} \Psi_r - \left( L_s - \frac{M^2}{L_r} \right) i_s \tag{7.43}$$

Substituting (7.43) into (7.37), and using the notation $x' = L_s - \frac{M^2}{L_r}$,

$$e_s = \frac{M}{L_r} \Psi_r - r_s i_s - x' \Psi I_s \tag{7.44}$$

Defining

$$e'_D = -\frac{M}{L_r} \psi_{rQ} \tag{7.45}$$

$$e'_Q = \frac{M}{L_r} \psi_{rD} \tag{7.46}$$

we obtain, from (7.44)

$$e_D = e'_D + x' i_Q - r_s i_D \tag{7.47}$$

$$e_Q = e'_Q - x' i_D - r_s i_Q \tag{7.48}$$

In the above equations we have dropped the subscript $s$ in voltages and currents for clarity.

From (7.42) and (7.38)

$$\frac{1}{\omega_o} \frac{d}{dt} \frac{L_r M}{r_s L_r} = -s W \Psi_r - r_r \Psi_r - \frac{M r_r}{L_r} i_s$$

Multiplying both sides by $\left( \frac{L_r M}{r_s L_r} \right)$, using (7.45), (7.46), and defining $T'_o = \frac{L_r}{\omega_o r_r}$

$$T'_o \frac{d}{dt} \begin{bmatrix} e'_Q \\ e'_D \end{bmatrix} = -s \omega_o T'_o \begin{bmatrix} e'_Q \\ e'_D \end{bmatrix} - \begin{bmatrix} e'_Q \\ e'_D \end{bmatrix} - (L_s - x') \begin{bmatrix} i_s \\ s \end{bmatrix}$$

or

$$T'_o \frac{d}{dt} \begin{bmatrix} e'_Q \\ e'_D \end{bmatrix} = -s \omega_o T'_o \begin{bmatrix} e'_D \\ e'_Q \end{bmatrix} - \begin{bmatrix} e'_Q \\ e'_D \end{bmatrix} - (x - x') \begin{bmatrix} i_s \\ s \end{bmatrix} \tag{7.49}$$

where $L_s = x$
Equation (7.49) can be expressed as separate equations

\[ T_o' \frac{de'_Q}{dt} = -e'_Q - (x - x')i_D - T_o' \omega_o s e'_Q \] (7.50)

\[ T_o' \frac{de'_D}{dt} = -e'_D + (x - x')i_Q + T_o' \omega_o s e'_Q \] (7.51)

Using (7.37), (7.47), (7.48), the torque expression (7.41) can be written as

\[ T_e = e'_D i_D + e'_Q i_Q \] (7.52)

As in the case of synchronous machine, \( e_D, e_Q, i_D, i_Q \), etc. can be treated as phasors, and (7.47), (7.48), and (7.50), (7.51), and (7.52) can be written as

\[ e_D + j e_Q = e'_D + j e'_Q - (r_s + jx')(i_D + j i_Q) \]

or

\[ \dot{e} = \dot{e}' - (r_s + jx') \dot{i} \] (7.53)

and

\[ T_o' \frac{d(e'_D + j e'_Q)}{dt} = -(e'_D + j e'_Q) - j(x - x')(i_D + j i_Q) - j T_o' \omega_o s (e'_D + j e'_Q) \]

or

\[ T_o' \frac{d\dot{e}'}{dt} = -\ddot{e}' - j(x - x') \dot{i} - j T_o' \omega_o s \dot{e}' \] (7.54)

\[ T_e = \Re(\dot{e}' \dot{i}^*) \] (7.55)

Figure 7.2 shows the equivalent circuit representing the transient behavior of induction motor.

Fig. 7.2 Equivalent circuit for transient behavior of induction motor

The equation of motion is

\[ \frac{2H}{\omega_o} \frac{d\omega}{dt} = T_m - T_e \]

where \( T_m \) is the mechanical torque

or

\[ 2H \frac{ds}{dt} = -(T_m - T_e) \] (7.56)

Note that for motor action both \( T_m \) and \( T_e \) are negative.
Reference 1 recommends the following expression for the mechanical torque

\[
\frac{T_m}{T_{mo}} = A \left( \frac{\omega_m}{\omega_{mo}} \right)^2 + B \left( \frac{\omega_m}{\omega_{mo}} \right) + C + D \left( \frac{\omega_m}{\omega_{mo}} \right)^{am} \tag{7.57}
\]

\[
C = 1 - (A + B + D) \tag{7.58}
\]

In the steady state \( \frac{d\hat{e}'}{dt} = 0 \), therefore, from (7.54),

\[
\hat{e}' = -\frac{j(x-x')}{1 + jT'_o \omega_o s} \omega \tag{7.59}
\]

Substituting (7.59) in (7.53), we obtain

\[
\frac{\hat{e}}{i} = -(r_s + jx') - \frac{j(x-x')}{1 + jT'_o \omega_o s} \tag{7.60}
\]

Equation (7.60) leads to the following equivalent circuit

In Figure 7.3, current direction is shown for motor action, for clarity.

The electrical torque

\[
T_e = \text{Re} (\hat{e}' i^*) = \text{Re} \left[ -\frac{j(x-x')}{1 + jT'_o \omega_o s} \hat{e}' i^* \right] = -\frac{(x-x')T'_o \omega_o s}{1 + (T'_o \omega_o s)^2} |i|^2
\]

From Figure 7.3

\[
\hat{i}_1 = -\hat{i}
\]

\[
\hat{i}_2 = -\frac{j(x-x')}{T'_o \omega_o s + j(x-x')}
\]

or

\[
\hat{i} = -\frac{1 + jT'_o \omega_o s}{jT'_o \omega_o s} \hat{i}_2
\]

\[
|\hat{i}|^2 = \frac{1 + (T'_o \omega_o s)^2}{(T'_o \omega_o s)^2} |\hat{i}_2|^2
\]

Fig. 7.3  Steady-state equivalent circuit of induction motor
and
\[ T_c = -i_2^2 \frac{x - x'}{T_o \omega_o s} = -i_2^2 \frac{r_2}{s} \]
where
\[ r_2 = \frac{x - x'}{T_o \omega_o} \]

As in synchronous machine, we can express \( x \) and \( x' \) in terms of mutual and leakage reactances
\[ x = x_s + x_m, \quad x' = x_s + \frac{x_m x_r}{x_m + x_r} \]
Also
\[ T_o' = \frac{x_m + x_r}{\omega_o r_r} \]

Therefore, an alternative form of the equivalent circuit is as shown in Figure 7.4. The derivation is left as an exercise.

![Fig. 7.4 An alternative form of equivalent circuit of induction motor](image)

**Initial condition calculation**

Given \( P, E \) (and \( Q \)) from power flow, we can calculate the initial slip, \( s \), as follows

From (7.60)
\[ \hat{i} = -\frac{1 + jT_o' \omega_o s}{r_s - T_o' \omega_o s x' + j(x + T_o' \omega_o s r_s)} \hat{e} \]
\[ P = \text{Re}(\hat{e}^* \hat{i}) = -\frac{r_s - T_o' \omega_o s x' + T_o' \omega_o s (x + T_o' \omega_o s r_s)}{(r_s - T_o' \omega_o s x')^2 + (x + T_o' \omega_o s r_s)^2} E^2 \quad (7.61) \]

The above yields two values of \( s \) (if a solution exists) for given values of \( P \) and \( E \). Alternatively, for a given value of \( E \), \( P \) can be calculated for assumed values of \( s \), starting with \( s = 0 \) (no-load condition). For \( s = 0 \),
\[ P = -\frac{r_s}{r_s^2 + x^2} E^2, \text{ the stator resistance loss} \]
\[ Q = -\text{Im}(\hat{e}^* \hat{i}) = -\frac{x + T_o' \omega_o s r_s - T_o' \omega_o s (r_s - T_o' \omega_o s x')}{(r_s - T_o' \omega_o s x')^2 + (x + T_o' \omega_o s r_s)^2} E^2 \quad (7.62) \]
For $s = 0$, \[ Q = -\frac{x}{r_s^2 + x^2} E^2 \]

The mismatch between $Q$ calculated above and $Q$ from power flow is adjusted by adding the required reactive compensation.

**Generic Load Model**

In conventional rotor angle -- large and small disturbance -- stability analyses, composite loads are usually represented by a ZIP or exponential model, although the IP part is not totally appropriate, as has been pointed out earlier. Large induction motor loads are often represented by detailed induction motor model, if their influence on the stability outcome is judged to be important. In voltage stability analyses, if the bulk of the load is induction motor, it should be represented as such, if the study results are going to be realistic. For loads other than motor loads, the use of the ZIP model, as is common in angle stability analyses, can produce grossly misleading results. If the load is known to be impedance load, and the impedance stays constant during the study period, it should be represented as such. In Chapter 10 it is shown that static constant $I$ and constant $P$ load models are inappropriate for voltage stability studies, and should therefore be avoided.

Voltage stability problems often evolve over a longer period of time than do the traditional angle stability problems. Many loads, e.g., thermostatically controlled heating loads, and loads controlled by under-load-tap-changing transformers and distribution voltage regulators tend to behave as constant power load over a longer term, even though the actual load itself is constant impedance. For example, space and water heaters will temporarily draw less power following a drop in voltage; however, they will stay on longer to generate the same amount of heat as would be produced at nominal voltage. Over a longer period the total energy drawn from the system would be, more or less, the same. An impedance load whose voltage is controlled would appear as constant power load to the system. After an initial drop in power consumption following a drop in voltage, these loads would tend to recover to their nominal level, i.e., collectively, they would behave as constant power load over a longer period of time, provided the control limits have not been reached. At limit, e.g., when the highest tap setting is reached, or the voltage is so low that the heating loads stay on continuously, these loads would behave as constant $Z$ load.

Various models have been proposed in the literature to reproduce the behavior of such loads in voltage stability studies. All these models are, however, flawed. Apparently, sufficient effort was not expended in understanding the underlying physical nature of the problem, as well as in the mathematical logic used [10 - 14] (see also Chapter 10). Also, in these models, active and reactive powers are modeled separately. In reality, there is always some coupling between active and reactive powers. It is not difficult to show that these models, in their stated forms, if valid, would be valid only in a specific range of parameter values, and when the right set of parameter values is selected, they are actually unnecessarily complex ways of representing the simplified first-order induction motor model.

Actually, the induction motor model, with appropriate parameter values, can faithfully represent any composite dynamic load, for both angle and voltage stability studies. The selection of the right parameter values may, however, present some challenge.
References


CHAPTER 8
SMALL DISTURBANCE STABILITY

Stability programs designed for large-disturbance (transient) stability studies simulate system response in time domain following a disturbance. The simulations are normally limited to a short duration, usually a few seconds of real time. If the generator rotors swing back before reaching a specified angle, the power system is considered large-disturbance stable. In the early days when electric power networks were relatively confined, and sophisticated control equipment were not generally utilized on the generators, the above criterion, also called “first swing stability,” was enough to assure eventual stability of the system against that particular disturbance. Instability in the initial period following a large disturbance is generally due to insufficient (or lack of) synchronizing torque between the interconnected generators. Automatic control equipment, e.g., fast acting excitation control, help improve the first swing stability by increasing the synchronizing torque. However, in the process they often reduce the damping torque, sometimes even rendering the overall damping negative, thereby causing oscillatory instability. With the growth of interconnection and application of advanced control equipment, consideration of proper damping of oscillations became more important. In a system capable of withstanding the initial shock of the disturbance, as evidenced by first swing stability, oscillations could continue at a reduced amplitude for a while, only to increase a few cycles later and eventually cause cascading line tripping and possibly system separation. This type of instability can manifest itself not only following a major disturbance but also following a sudden small change in system condition not generally classified as major disturbance, e.g., a moderate amount of load tripping, a sudden addition of a large load, tripping of a minor transmission line, etc.

In order for a power system to be operable, it must have an equilibrium point that is stable. This means it must be small-disturbance stable (see Chapter 2). In an interconnected system, due to an improper selection of control parameters, a stable equilibrium point may not exist at all. Oscillations once started can build up gradually although they may not be apparent during the first few cycles. On the other hand a well designed control system can extend the stability limit considerably.

Conventional stability programs designed for large-disturbance stability studies can, in principle, be used to study the above phenomenon by running the program for a much longer period of real time than is normally required to establish large-disturbance, first swing stability. This approach can, however, be impractical and the result may not be fully conclusive.

Immediately following a small disturbance, or following a large disturbance after the system has survived the initial shock (i.e., it’s first swing stable) and entered a state of oscillation, the system nonlinearities do not play a major role. The power system can therefore be linearized about the equilibrium point and useful information on the system small-disturbance performance can be obtained from the linearized model. This would permit efficient design of the control, allowing necessary fixes to combat instability. Certain aspects of the system stability problem are more readily detectable, and hence correctable, by studying the linearized system.

Although there are several methods of obtaining stability information from a linearized system, a state-space approach is desirable since in addition to providing all the information as obtainable
from other approaches this method provides certain additional advantages and can handle, with sufficient ease, complex systems.

After linearization of the equations representing the power system dynamics around an operating point, which will retain the behavior of the system under small disturbances, they can be reduced to the state-space form \( \dot{x} = Ax \), where \( x \) is an \( n \)-vector of the variables representing the state of the system, and \( A \) is a matrix of real constant coefficients, called the coefficient (or system) matrix. The solution and stability condition of this system has been discussed in detail in Chapter 2. Briefly, for asymptotic stability, the eigenvalues of \( A \) must have negative real parts. If there is an eigenvalue with positive real part the equilibrium is unstable. Eigenvalues of \( A \) will be either real or complex. If complex, they will occur in conjugate pairs. By studying these eigenvalues much of the information on the system performance can be obtained. For a stable system the larger the magnitude of the real part the faster the transient corresponding to that eigenvalue will disappear. Any increase in the magnitude of the real part would mean a corresponding increase in damping. The magnitude of the imaginary part indicates the frequency of oscillation of the transient.

Interconnected generators, due to the inertias of the rotors, act as spring-mass systems, and as a result a power system is inherently oscillatory (see Chapter 3). Two types of oscillations are generally noticeable. When there is a disturbance in an area with closely coupled generators, the machine rotors swing against each other and with respect to the rest of the system. This is the local mode of oscillation. The special case is a single generating unit transmitting power over a long transmission line to a load center, and the generator rotor swings against the rest of the system. The characteristic local mode frequency is, in general, in the 0.5 - 2 Hz range, depending on the machine and system parameters and loading conditions.

In large interconnected systems closely coupled generators in one area are connected through tie-lines to other areas with closely coupled generators. Following a disturbance in the tie-line flow, generators in one area (swinging more or less in unison) swing against generators in other areas (again, swinging more or less in unison). This is the inter-area mode. This mode of oscillation can also appear following a major disturbance in any of the areas separated by tie-lines. The characteristic frequency of the inter-area mode of oscillation is generally in the range of 0.1 - 0.5 Hz. The range is lower than the corresponding range for local mode because of higher combined inertias and relatively weak tie-lines. Both local and inter-area modes tend to be coupled to some degree. Although it is possible to identify these modes in time domain simulation, it can often be done more conveniently from an analysis of the linearized system, and any coupling between modes can also be revealed.

Due to the excessive time involved in obtaining complete eigensolutions of large matrices, special methods have been developed which aim to compute a selected number of eigenvalues and the associated eigenvectors in studies of very large systems. In one method a limited number of eigenvalues associated with rotor angle modes are computed. In several other methods eigenvalues associated with a small number of selected modes of interest are computed. These methods can handle very large systems by taking advantage of network sparsity.

All these methods, however, have major drawbacks. Frequently they fail to identify and compute the critical mode(s). Note that instability can be caused by any mode, not necessarily a particular rotor angle mode or any other mode that might appear to be a likely candidate. In a complex system it is not a simple matter to identify such modes a priori. As can be seen from the
expression for the solution of the linearized system (eqn. 2.43 of Chapter 2), depending on the initial state (the system state at the end of the assumed small disturbance), instability can be caused by any mode out of several thousands in a large system. An evaluation of the complete set of eigenvalues (modes) is therefore needed. For this purpose a large system can be reduced to manageable size. When the interest lies locally, distant areas can be replaced by equivalents. For inter-area oscillation studies, individual areas can be reduced in size by combining the machines that swing in coherent groups.

Actually, by judiciously choosing the modeling details, a complete eigenvalue analysis of fairly large systems can be handled by today’s computing facilities. To accomplish this, machines in the immediate vicinity of the area under investigation can be represented in full detail. Machines outside and adjacent to the area can be represented by the classical model, and the rest of the system can be represented by equivalents.

**State Space Representation of Power System**

A power system can be represented by the following sets of differential and algebraic equations:

\[
\dot{x} = f(x, y) \quad 0 = g(x, y) \tag{8.1}
\]

After linearization the above can be expressed as

\[
\begin{bmatrix}
\Delta \dot{x} \\
0
\end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{8.2}
\]

From (8.2)

\[
\Delta \dot{x} = [A_1 - A_2 A_4^{-1} A_3] \Delta x = A \Delta x \tag{8.3}
\]

**Single machine-infinite bus system**

Considering synchronous machine model 2 (see Chapter 5), the differential equations for the machine internal voltages are given by

\[
T_{pq}' \frac{de_q'}{dt} = (x_q' - x_q')i_q - e_q' \tag{8.4}
\]

\[
T_{dq}' \frac{de_d'}{dt} = E_{fd} - (x_d' - x_d')i_d - e_d' \tag{8.5}
\]

The machine voltage equations:

\[
e_d = e_d' - r i_d + x_q'i_q \\
e_q = e_q' - r i_q - x_d'i_d \tag{8.6}
\]

The equation of motion:

\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} = T_m - T_e - \frac{K_d}{\omega_o} (\omega - \omega_o) \tag{8.7}
\]

\[
T_e = e_d'i_d + e_q'i_q + (x_q' - x_d')i_d i_q \tag{8.8}
\]
For the transmission system (see Fig. 6.4 of Chapter 6)
\[ V_b \sin \delta = e_d - r_i i_d + x_c i_q \]
\[ V_b \cos \delta = e_q - r_i i_q - x_c i_d \]

(8.9)

Linearizing equations (8.4) - (8.9)
\[ T_q' \frac{d\Delta e_d'}{dt} = (x_q - x_q') \Delta i_q - \Delta e_d' \]
(8.10)
\[ T_d' \frac{d\Delta e_q'}{dt} = \Delta E_{fd} - (x_d - x_d') \Delta i_d - \Delta e_q' \]
(8.11)
\[ \Delta e_d = \Delta e_d' - r \Delta i_d + x'_q \Delta i_q \]
\[ \Delta e_q = \Delta e_q' - r \Delta i_q - x'_q \Delta i_d \]
(8.12)
\[ 2H_0 \frac{d^2 \Delta \delta}{dt^2} = -\Delta T_e - K_d \Delta \omega \]
(8.13)
\[ \Delta T_e = \Delta e_d' i_d + e_d' i_q + \Delta e_q' i_q + \Delta e_d q + (x'_q - x_q) (\Delta i_d i_q + i_d i_q) \]
\[ = \Delta e_d' i_d + \Delta e_q' i_q + \left[ e_d' + (x'_q - x_q) i_q \right] \Delta i_d + \left[ e_q' + (x'_q - x_q) i_d \right] \Delta i_q \]
(8.14)

We will assume constant mechanical torque -- \( \Delta T_m = 0 \) (i.e., no governor action). Inclusion of governor control is quite straightforward.
\[ V_b \cos \delta \Delta \delta = \Delta e_d - r_c \Delta i_d + x_c \Delta i_q \]
\[ -V_b \sin \delta \Delta \delta = \Delta e_q - r_c \Delta i_q - x_c \Delta i_d \]
(8.15)

Considering the simplified excitation control system used in Chapter 6 (Fig. 6.6, IEEE type 1), ignoring saturation for the purpose of this illustration, the linearized equations can be written in matrix form as (see Appendix C)
\[
\frac{d}{dt} \begin{bmatrix} \Delta E_{fd} \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{K_p K_d}{T_p' T_A} + \frac{1}{T_A} & K_p K_d \\ \frac{1}{T_p} & -\frac{1}{T_p} \end{bmatrix} \begin{bmatrix} \Delta E_{fd} \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{K_d}{T_A} \\ 0 \end{bmatrix} \Delta V_t
\]
(8.16)

Since \( V_t^2 = e_d^2 + e_q^2 \), \( \Delta V_t \) can be expressed as
\[
\Delta V_t = \begin{bmatrix} e_d \\ \frac{e_q}{V_t} \end{bmatrix} \begin{bmatrix} \Delta e_d \\ \Delta e_q \end{bmatrix}
\]
(8.17)

Note that in equations (8.14) - (8.17) \( i_d, i_q, e_d, e_q, e'_d, e'_q, \delta, V_t \) are all values at the initial operating point, obtained from power flow.

Equations (8.10) - (8.17) can be arranged in the form of (8.2), where
\[ \Delta x = \begin{bmatrix} \Delta \delta & \Delta \omega & \Delta e'_d & \Delta e'_q & \Delta E_{jd} & \Delta x_2 \end{bmatrix} \]
\[ \Delta y = \begin{bmatrix} \Delta i_d & \Delta i_q & \Delta e_d & \Delta e_q \end{bmatrix} \]
\[ A_1 = \begin{bmatrix} 1 & -\frac{K_d}{2H} & -\frac{\omega_o}{2H} & -\frac{\omega_o}{2H} & \frac{1}{T_{qo}} & -1 \\ \frac{1}{T_{do}} & -\frac{1}{T_{do}} & -\frac{1}{T_{do}} & -\frac{1}{T_{do}} & \frac{1}{T_{f}} & \frac{K_fK_A}{T_F} \\ \frac{1}{T_{f}} & \frac{1}{T_{f}} & \frac{1}{T_{f}} & \frac{1}{T_{f}} & -1 \\ -\frac{\omega_o}{2H} & -\frac{\omega_o}{2H} & \frac{x_q - x'_q}{T_{qo}} & \frac{x_d - x'_d}{T_{do}} & -\frac{K_A e_d}{T_A V_t} & -\frac{K_A e_q}{T_A V_t} \end{bmatrix} \]
\[ A_2 = \begin{bmatrix} \frac{1}{T_{do}} & -1 & -1 \\ V_b \cos \delta & -V_b \sin \delta \end{bmatrix}, \quad A_4 = \begin{bmatrix} r & -x'_q & 1 \\ x'_d & r & 1 \\ r_c & -x_c & -1 \\ x_c & r_c & -1 \end{bmatrix} \]

from which the state model follows.

**State model of the multi-machine system**

In a multi-machine system using 2-axis representation the relationship between the individual machine terminal voltages (or currents) in terms of the machine reference frame and network reference frame is given by equations of the form of equations (5.182) and (5.183) of Chapter 5. For a system containing \( n \) synchronous machines the voltage relationship is
\[
\begin{bmatrix}
e_{d1} \\
e_{q1} \\
e_{d2} \\
e_{q2} \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\sin \delta_1 & -\cos \delta_1 \\
\cos \delta_1 & \sin \delta_1 \\
\sin \delta_2 & -\cos \delta_2 \\
\cos \delta_2 & \sin \delta_2 \\
\vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
e_{d1} \\
e_{q1} \\
e_{d2} \\
e_{q2} \\
\vdots
\end{bmatrix}
\]

or
\[
e = TV
\]

Similarly,
\[
i = TI
\]

Linearizing (8.18),
\[
\Delta e = T \Delta V + \Delta TV = T \Delta V + M \Delta \delta
\]

where
\[
M = \begin{bmatrix}
e_{q1} & -e_{d1} & e_{q2} & -e_{d2} & \cdots \\
\end{bmatrix}, \quad \Delta \delta = \begin{bmatrix}
\Delta \delta_1 \\
\Delta \delta_2 \\
\vdots
\end{bmatrix}
\]

From (8.20) we can also write
\[
\Delta V = T^{-1} \Delta e - T^{-1} M \Delta \delta = T^{-1} \Delta e + N \Delta \delta
\]

where
\[
N = -T^{-1} M = \begin{bmatrix}
-e_{q1} & e_{d1} \\
e_{q2} & -e_{d2} \\
\vdots & \vdots
\end{bmatrix}
\]

Similarly, from (8.19),
\[
\Delta i = T \Delta I + \overline{M} \Delta \delta
\]

and
\[
\Delta I = T^{-1} \Delta i + \overline{N} \Delta \delta
\]

where
\[
\overline{M} = \begin{bmatrix}
i_{q1} & -i_{d1} & i_{q2} & -i_{d2} & \cdots \\
\end{bmatrix}, \quad \overline{N} = -T^{-1} \overline{M} = \begin{bmatrix}
-i_{q1} & i_{d1} \\
i_{q2} & -i_{d2} \\
\vdots & \vdots
\end{bmatrix}
\]
The relationship between the machine terminal voltages and currents in network reference frame, in linearized form, is given by

\[ \Delta I = Y_N \Delta V \]  

(8.24)

where \( Y_N \) is the \( 2n \times 2n \) network admittance matrix written in real form after eliminating the non-machine nodes and then separating the real and imaginary parts (see Chapter 5).

Using (8.23) and (8.21), (8.24) can be written as

\[ T^{-1} \Delta i + \bar{N} \Delta \delta = Y_N \left[ T^{-1} \Delta e + N \Delta \delta \right] \]

from which

\[ \Delta i = T Y_N T^{-1} \Delta e + \left[ \bar{M} + T Y_N N \right] \Delta \delta \]  

(8.25)

Equation (8.25) can also be written as

\[ \Delta e = T Y_N^{-1} T^{-1} \Delta i + \left[ M + T Y_N^{-1} N \right] \Delta \delta \]  

(8.26)

For the multi-machine case we could write the differential equations in matrix form as

\[ \Delta \dot{x}_i = A_{i1} \Delta x_i + A_{i2} \Delta y_i \quad i = 1, 2, \ldots n \]  

(8.27)

the machine algebraic equations (8.12) as

\[ 0 = -\Delta E_i + Z_{Mi} \Delta I_i + \Delta e_i \quad i = 1, 2, \ldots n \]  

(8.28)

where

\[ \Delta E_i = \begin{bmatrix} \Delta e_{di} \\ \Delta e_{qi} \end{bmatrix}, \quad Z_{Mi} = \begin{bmatrix} r_i & -x'_{qi} \\ x'_{di} & r_i \end{bmatrix} \]

and the network equations (8.25) as

\[ 0 = \left[ \bar{M} + T Y_N N \right] \Delta \delta - \Delta i + T Y_N T^{-1} \Delta e \]  

(8.29)

Equations (8.27) - (8.29) could be arranged in the form of (8.2), from which the state model could be obtained as shown in (8.3).

Note that most of the submatrices involved are block diagonal and sparse except for \( \bar{M} + T Y_N N \) and \( T Y_N^{-1} T^{-1} \) which are full. However, the method is cumbersome for large systems. A method that is more suitable for multi-machine systems is as follows:

For a group of machines (8.27) can be written as

\[ \Delta \dot{x} = A_{i1} \Delta x + B \Delta I_i + C \Delta e \]  

(8.30)

where

\[ \Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \end{bmatrix}, \quad \Delta I_i = \begin{bmatrix} \Delta i_1 \\ \Delta i_2 \\ \vdots \end{bmatrix}, \quad \Delta e = \begin{bmatrix} \Delta e_1 \\ \Delta e_2 \\ \vdots \end{bmatrix} \]

The machine algebraic equations can be grouped as

\[ \Delta I_i = Z_{M_i}^{-1} \Delta E - Z_{M_i}^{-1} \Delta e \]
where
\[
\Delta E = \begin{bmatrix} \Delta E_1 \\ \Delta E_2 \\ \vdots \end{bmatrix}, \quad Z_M^{-1} = \begin{bmatrix} Z_{M1}^{-1} & \cdots \\ \cdots & \cdots \end{bmatrix}
\]
and can be expressed as
\[
\Delta i = D\Delta x - Z_M^{-1}\Delta e
\]
(8.31)
where \(D\) is obtained by augmenting \(Z_M^{-1}\) with zeros.

Similarly, the network equations (8.25) can be expressed as
\[
\Delta i = G\Delta x + Y_N T^{-1} \Delta e
\]
(8.32)
Using (8.31), (8.30) reduces to
\[
\Delta x = [A_1 + BD]\Delta x + [C - B Z_M^{-1}]\Delta e
\]
(8.33)
From (8.31) and (8.32)
\[
[T Y_N T^{-1} + Z_M^{-1}]\Delta e = [D - G]\Delta x
\]
from which
\[
\Delta e = T[Y_N + T^{-1}Z_M^{-1}T]^{-1}T^{-1}[D - G]\Delta x
\]
(8.34)
Substituting (8.34) in (8.33) we obtain
\[
\Delta x = A\Delta x
\]
where
\[
A = A_1 + BD + [C - B Z_M^{-1}]T[Y_N + T^{-1}Z_M^{-1}T]^{-1}T^{-1}[D - G]
\]
The formation of the coefficient matrix can be further simplified by grouping the state variables and forming the state equations for each group, and then combining them together. The method has the advantage of requiring the least amount of computation and storage space. It also makes it simpler to locate the various system and control parameters in the final matrix, thus facilitating eigenvalue sensitivity analyses.

**An Efficient Method of Deriving the State Model**

The method will first be illustrated for a synchronous machine model where the various flux linkages are among the state variables. The manipulation of the equations in the intermediate stages before the final coefficient matrix is formed is somewhat simpler for this model than for models 1, 2, or 3 (see Chapter 5).

**State model using flux linkages as state variables**

The relevant equations for the synchronous machine, assuming one damper winding on each axis, are listed below.
SMALL DISTURBANCE STABILITY

Flux linkage equations:

\[
\begin{align*}
\psi_d &= -x_d i_d + x_{ad} i_{fd} + x_{ad} i_{ld} \\
\psi_q &= -x_q i_q + x_{aq} i_{1q} \\
\psi_{fd} &= -x_{ad} i_d + x_{fd} i_{fd} + x_{f1d} i_{ld} \\
\psi_{1d} &= -x_{ad} i_d + x_{f1d} i_{fd} + x_{11d} i_{ld} \\
\psi_{1q} &= -x_{aq} i_q + x_{11q} i_{1q}
\end{align*}
\]  

(8.35)

Voltage equations:

\[
\begin{align*}
e_d &= \frac{1}{\omega_o} \frac{d\psi_d}{dt} - \frac{\omega}{\omega_o} \psi_q - ri_d \\
e_q &= \frac{1}{\omega_o} \frac{d\psi_q}{dt} + \frac{\omega}{\omega_o} \psi_d - ri_q \\
e_{fd} &= \frac{1}{\omega_o} \frac{d\psi_{fd}}{dt} + r_{fd} i_{fd} \\
0 &= \frac{1}{\omega_o} \frac{d\psi_{1d}}{dt} + r_{1d} i_{1d} \\
0 &= \frac{1}{\omega_o} \frac{d\psi_{1q}}{dt} + r_{1q} i_{1q}
\end{align*}
\]  

(8.36)

Air-gap torque:

\[T_e = \psi_d i_q - \psi_q i_d\]  

(8.37)

The flux linkage equations in linearized form can be written in matrix form as

\[
\begin{bmatrix}
\Delta \Psi_s \\
\Delta \Psi_r
\end{bmatrix} = X \begin{bmatrix}
\Delta i \\
\Delta i_r
\end{bmatrix}
\]  

(8.38)

where

\[
\Delta \Psi_s = \begin{bmatrix}
\Delta \psi_d \\
\Delta \psi_q
\end{bmatrix}, \quad \Delta \Psi_r = \begin{bmatrix}
\Delta \psi_{fd} \\
\Delta \psi_{1d} \\
\Delta \psi_{1q}
\end{bmatrix}, \quad \Delta i = \begin{bmatrix}
\Delta i_d \\
\Delta i_q
\end{bmatrix}, \quad \Delta i_r = \begin{bmatrix}
\Delta i_{fd} \\
\Delta i_{1d} \\
\Delta i_{1q}
\end{bmatrix}
\]

and

\[
X = \begin{bmatrix}
-x_d & x_{ad} & x_{ad} \\
-x_q & -x_{aq} & \\
-x_{ad} & x_{fd} & x_{f1d} \\
-x_{ad} & x_{f1d} & x_{11d} \\
-x_{aq} & & x_{11q}
\end{bmatrix}
\]
From (8.38)

\[
\begin{bmatrix}
\frac{\Delta i}{\Delta r}
\end{bmatrix} = Y \begin{bmatrix}
\frac{\Delta \Psi}{\Delta r}
\end{bmatrix}
\]

(8.39)

where \( Y \) is the inverse of \( X \).

Equation (8.39) can be written as

\[
\begin{bmatrix}
\frac{\Delta i}{\Delta r}
\end{bmatrix} = Y \begin{bmatrix}
\frac{Y_{11}}{Y_{12}} + \frac{Y_{21}}{Y_{22}}
\end{bmatrix} \begin{bmatrix}
\Delta \Psi
\end{bmatrix}
\]

For \( n \) machines the above can be split up as

\[
\begin{bmatrix}
\Delta i_1 \\
\Delta i_2 \\
\vdots
\end{bmatrix} = \begin{bmatrix}
Y_{11}^i & Y_{12}^i \\
Y_{21}^i & Y_{22}^i
\end{bmatrix} \begin{bmatrix}
\Delta \Psi_1 \\
\Delta \Psi_2 \\
\vdots
\end{bmatrix}
\]

where

\[
Y_i^i = \begin{bmatrix}
Y_{11}^i \\
Y_{12}^i
\end{bmatrix} \quad \text{and} \quad \Delta \Psi_i = \begin{bmatrix}
\Delta \Psi_1^i \\
\Delta \Psi_2^i
\end{bmatrix} \quad i = 1, 2, \ldots, n
\]

or

\[
\Delta i = P \Delta \Psi
\]

(8.40)

Similarly,

\[
\Delta i_r = Q \Delta \Psi
\]

(8.41)

where

\[
Q = \begin{bmatrix}
Y_2^1 & Y_2^2 \\
Y_2^2 & \ddots
\end{bmatrix} \quad \text{and} \quad Y_i^2 = \begin{bmatrix}
Y_{11}^i \\
Y_{12}^i
\end{bmatrix} \quad i = 1, 2, \ldots, n
\]

The machine voltage equations (8.36), after linearization and assuming \( \omega \approx \omega_o \), can be written in matrix form as

\[
\begin{bmatrix}
\Delta \psi_d \\
\Delta \psi_q \\
\Delta \psi_{fd} \\
\Delta \psi_{qd}
\end{bmatrix} = \begin{bmatrix}
\omega_o & -\omega_o & 0 & 0 \\
0 & \omega_o & 0 & -\omega_o \\
-\omega_o r_{fd} & 0 & \omega_o r_{fd} & -\omega_o r_{qd} \\
0 & -\omega_o r_{qd} & \omega_o r_{qd} & -\omega_o r_{fd}
\end{bmatrix} \begin{bmatrix}
\Delta i_d \\
\Delta i_q \\
\Delta i_{fd} \\
\Delta i_{qd}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\omega_o & -\omega_o & 0 & 0 \\
0 & \omega_o & 0 & -\omega_o \\
-\omega_o r_{fd} & 0 & \omega_o r_{fd} & -\omega_o r_{qd} \\
0 & -\omega_o r_{qd} & \omega_o r_{qd} & -\omega_o r_{fd}
\end{bmatrix} \begin{bmatrix}
\Delta e_d \\
\Delta e_q \\
x_{fd} \frac{T_{do}}{x_{ad}}
\end{bmatrix} \Delta E_{fd}
\]

8-10
In the above we have used the relationships 
\[ E_{fd} = \frac{x_{ad}}{r_{fd}} e_{fd} \] 
and 
\[ T'_{do} = -\frac{x_{fd}}{\omega_{o} r_{fd}}. \]

Using (8.39) the above can be written as
\[ \Delta \dot{\Psi} = S \Delta \Psi + W \Delta e + R \Delta x_{ex} \]  \hspace{1cm} (8.42)

where

\[ R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{x_{fd}}{x_{ad} T'_{do}} & 0 \\ \frac{x_{ad} T'_{do}}{0} & 0 \end{bmatrix}, \quad \Delta x_{ex} = \begin{bmatrix} \Delta E_{fd} \\ \Delta x_2 \end{bmatrix} \]

For \( n \) machines,
\[ S = \begin{bmatrix} S_1 & \cdots & \cdots & \cdots \\ S_2 & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & \cdots & \cdots & \cdots \\ W_2 & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \]

Using (8.26) and (8.40), (8.42) becomes
\[ \Delta \dot{\Psi} = S \Delta \Psi + W Y^{-1} T^{-1} P \Delta \Psi + W \left[ M + T Y^{-1} N \right] \Delta \delta + R \Delta x_{ex} \]
\[ = B \Delta \Psi + C \Delta \delta + R \Delta x_{ex} \]  \hspace{1cm} (8.43)

where
\[ B = S + W Y^{-1} T^{-1} P, \quad C = W \left[ M + T Y^{-1} N \right] \]

The equation of motion in linearized form, with \( \Delta T_m = 0 \), is given by (8.13), where
\[ \Delta T_e = \psi_d \Delta i_q - \psi_q \Delta i_d + i_q \Delta \psi_d - i_d \Delta \psi_q \]

For \( n \) machines the equation of motion can be expressed as
\[ \Delta \dot{\delta} = \Delta \omega \]  \hspace{1cm} (8.44)
\[ \Delta \omega = i \Delta \Psi + \Psi \Delta i + K_d \Delta \omega \]  \hspace{1cm} (8.45)

where
\[ \Delta \omega = \begin{bmatrix} \Delta \omega_1 \\ \Delta \omega_2 \\ \vdots \end{bmatrix}, \quad i = \begin{bmatrix} -\frac{\omega_o}{2H_1} i_{q1} & \frac{\omega_o}{2H_1} i_{d1} & 0 & 0 & 0 \\ -\frac{\omega_o}{2H_2} i_{q2} & \frac{\omega_o}{2H_2} i_{d2} & 0 & 0 & 0 \end{bmatrix} \]
\[
\Psi = \begin{bmatrix}
\frac{\omega_0}{2H_1} \psi_{q1} & -\frac{\omega_0}{2H_1} \psi_{d1} \\
\frac{\omega_0}{2H_2} \psi_{q2} & \frac{\omega_0}{2H_2} \psi_{d2} \\
\ddots & \ddots
\end{bmatrix}
\]

\[K_d = \text{diag} \begin{bmatrix}
-\frac{K_{d1}}{2H_1} & -\frac{K_{d2}}{2H_2} \\
\ddots & \ddots
\end{bmatrix}\]

Using (8.40), (8.45) becomes

\[
\Delta \dot{\omega} = i \Delta \Psi + \Psi P \Delta \Psi + K_d \Delta \omega
\]

where

\[F = i + \Psi P\]

From (8.16) and (8.17) the linearized equations for the excitation control is

\[
\Delta \dot{x}_{ex} = \begin{bmatrix} K_{ex} \end{bmatrix} \Delta x_{ex} + L \Delta e
\]

where

\[
K_{ex} = \begin{bmatrix}
-K_f K_A & K_f K_A \\
\frac{1}{T_y} + \frac{1}{T_A} & -\frac{1}{T_y} \\
\frac{1}{T_f} & -\frac{1}{T_f}
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
-K_A e_d & K_A e_q \\
0 & 0
\end{bmatrix}
\]

For \(n\) machine system

\[
K_{ex} = \begin{bmatrix}
K_{ex1} & \ddots \\
K_{ex2} & \ddots \\
\ddots & \ddots
\end{bmatrix}, \quad L = \begin{bmatrix}
L_1 & \ddots \\
L_2 & \ddots
\end{bmatrix}
\]

Using (8.26) and (8.40), (8.47) becomes

\[
\Delta \dot{x}_{ex} = K_{ex} \Delta x_{ex} + L \begin{bmatrix} T Y^{-1} \end{bmatrix} \Delta i + L \left[ M + T Y^{-1} \right] \Delta \delta
\]

\[
= K_{ex} \Delta x_{ex} + G \Delta \Psi + H \Delta \delta
\]

where

\[G = L \begin{bmatrix} T Y^{-1} \end{bmatrix} \] and \[H = L \left[ M + T Y^{-1} \right] \]
From (8.44), (8.46), (8.43) and (8.48) the state equation can be written as
\[
\begin{bmatrix}
\Delta \dot{\delta} \\
\Delta \dot{\omega} \\
\Delta \dot{\Psi} \\
\Delta \dot{x}_{ex}
\end{bmatrix} =
\begin{bmatrix}
I & F \\
K_d & 0 \\
C & B & R \\
H & G & K_{ex}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta \\
\Delta \omega \\
\Delta \Psi \\
\Delta x_{ex}
\end{bmatrix} +
\begin{bmatrix}
\Delta \delta \\
\Delta \omega \\
\Delta \Psi \\
\Delta x_{ex}
\end{bmatrix}
\] (8.49)

Note that most of the matrices and submatrices involved are block diagonal and sparse and this can be taken advantage of in the computations.

The state model (8.49) is not unique. The state variables can be ordered in many other ways by reordering the coefficient (A) matrix accordingly.

**State model for machine model 2**

From (8.25) and (8.28)
\[
Z^{-1}_M \Delta E - Z^{-1}_M \Delta e = TY_N T^{-1} \Delta e + [\bar{M} + TY_N N] \Delta \delta
\]
from which
\[
\Delta e = T \left[ Y_N + T^{-1}Z^{-1}_M T \right]^{-1} T^{-1}Z^{-1}_M \Delta E - T \left[ Y_N + T^{-1}Z^{-1}_M T \right]^{-1} T^{-1} \left[ \bar{M} + TY_N N \right] \Delta \delta
\] (8.50)

Substituting (8.50) in (8.28)
\[
\Delta i = A \Delta E + B \Delta \delta
\] (8.51)
where
\[
A = Z^{-1}_M - Z^{-1}_M T \left[ Y_N + T^{-1}Z^{-1}_M T \right]^{-1} T^{-1}Z^{-1}_M
\]
\[
B = Z^{-1}_M T \left[ Y_N + T^{-1}Z^{-1}_M T \right]^{-1} T^{-1} \left[ \bar{M} + TY_N N \right]
\]

The linearized differential equations for the machine internal voltages, (8.10) and (8.11), can be written in matrix form as
\[
\begin{bmatrix}
\Delta \dot{e}_d \\
\Delta \dot{e}_q
\end{bmatrix} =
\begin{bmatrix}
-1/T'_{qo} & \frac{x_q - x'_q}{T'_{qo}} \\
-1/T'_{do} & -\frac{x_d - x'_d}{T'_{do}}
\end{bmatrix}
\begin{bmatrix}
\Delta e'_d \\
\Delta e'_q
\end{bmatrix} +
\begin{bmatrix}
\frac{1}{T'_{qo}} & 0 \\
0 & \frac{1}{T'_{do}}
\end{bmatrix}
\begin{bmatrix}
\Delta i_d \\
\Delta i_q
\end{bmatrix}
\]
\[
\begin{bmatrix}
\Delta E_{fd}
\end{bmatrix} =
\begin{bmatrix}
\Delta E_{fd}
\end{bmatrix}
\]
As before, for n machines the above can be written as
\[
\Delta \tilde{E} = P \Delta E + X \Delta i + Q \Delta x_{ex}
\]
which, using (8.51), becomes
\[
\Delta \tilde{E} = C \Delta \delta + D \Delta E + Q \Delta x_{ex}
\] (8.52)
where
\[
C = XB, \quad D = P + XA
\]

From (8.13) and (8.14), the linearized equations of motion for n machines can be written in matrix form as
\[ \Delta \dot{\delta} = \Delta \omega \]  
\[ \Delta \dot{\omega} = i \Delta E + e \Delta i + K_d \Delta \omega \]  

where

\[
i = \begin{bmatrix}
-\frac{\omega_0}{2H_1} i_{d1} & -\frac{\omega_0}{2H_1} i_{q1} \\
-\frac{\omega_0}{2H_2} i_{d2} & -\frac{\omega_0}{2H_2} i_{q2}
\end{bmatrix}
\]

\[
e = \begin{bmatrix}
-\frac{\omega_0}{2H_1} \left( e'_{d1} + x'_{q1} i_{q1} \right) + \frac{\omega_0}{2H_1} \left( e'_{q1} i_{q1} \right) \\
-\frac{\omega_0}{2H_2} \left( e'_{d2} + x'_{q2} i_{q2} \right) - \frac{\omega_0}{2H_2} \left( e'_{q2} i_{q2} \right)
\end{bmatrix}
\]

Using (8.51), (8.54) becomes

\[ \Delta \dot{\omega} = F \Delta E + J \Delta \delta + K_d \Delta \omega \]  

where

\[ F = i + eA, \quad J = eB \]

Excitation control equations (8.47) can be written as, using (8.50),

\[ \Delta \dot{x}_{ex} = K_{ex} \Delta x_{ex} + G \Delta E + H \Delta \delta \]  

where

\[ G = LT \left[ Y_N + T^{-1} Z_M^{-1} \right]^{-1} T^{-1} Z_M^{-1} \]

\[ H = -LT \left[ Y_N + T^{-1} Z_M^{-1} \right]^{-1} T^{-1} \left[ M + T Y_N N \right] \]

From (8.53), (8.55), (8.52) and (8.56) the state model follows

\[ \begin{bmatrix} \Delta \dot{\delta} \\ \Delta \dot{\omega} \\ \Delta \dot{E} \\ \Delta \dot{x}_{ex} \end{bmatrix} = \begin{bmatrix} I & J & K_d & F \\ J & K & D & Q \\ C & D & G & K_{ex} \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta E \\ \Delta x_{ex} \end{bmatrix} \]

**Mixed machine models**

In multi-machine studies it is often desirable to employ different machine models to represent the various machines in the system. For example, if one or several portions of the system are represented by equivalent machines they may be represented by constant voltages behind transient reactances, the so-called classical model, while the more detailed representations are
reserved only for the machines whose dynamic characteristics are desired to be accurately evaluated. Also, in a system there may be a number of large induction motors which may greatly affect the dynamic characteristics of the system and hence a detailed representation of these may be desirable.

For the purpose of illustration the formation of the coefficient matrix for a system where some of the synchronous machines are represented by model 2 and the rest by classical model will be considered. It will however be clear that any combination of machine and control system representation can similarly be handled.

The equations for the classical model can be written in the 2-axis representation as

\[ e_d = -r i_d + x'_d i_q \]
\[ e_q = e'_q - r i_q - x'_d i_d \]

where \( e'_q \) is the constant voltage behind transient reactance \( x'_d \). This voltage being constant there will be no differential equation to represent variation of flux linkages.

After linearization, the voltage equations can be written in matrix form, for \( n \) machines, as

\[ \Delta E = Z \Delta i + \Delta e \] (8.58)

where \( \Delta E \) is a null vector.

For the complete system (8.28) and (8.58) are written as one matrix equation before forming (8.51).

The expression for the electrical torque is

\[ T_e = e'_q i_q \]

which after linearization becomes

\[ \Delta T_e = e'_q \Delta i_q \]

Therefore the linearized equation of motion for the machines employing the classical model, in matrix form

\[ \Delta \dot{\omega} = e \Delta i + K_d \Delta \omega \] (8.59)

where

\[ e = \begin{bmatrix} 0 & -\frac{\omega_o}{2H_1} e'_q \\ -\frac{\omega_o}{2H_2} e'_q & 0 \end{bmatrix} \]

For the complete system, equations (8.54) and (8.59) are written as one matrix equation of the form of (8.54), the matrix \( i \) being augmented with the appropriate number of zeros, while the new \( e \) matrix is given by
As before, by eliminating $\Delta i$ the equation reduces to that given by (8.55).

**State model for systems containing induction motor loads**

The equations for the induction motor are given in Chapter 7 and reproduced here for convenience.

\[
T'_d \frac{de_D'}{dt} = -e' + (x-x')i_Q + T'_o \omega_s e_Q' \quad (8.60)
\]

\[
T'_o \frac{de_Q'}{dt} = -e' - (x-x')i_D - T'_o \omega_s e_D' \quad (8.61)
\]

\[
e_D = e'_D + x'i_Q - ri_D \\ e_Q = e'_Q - x'i_D - ri_Q \\ T_c = e'_D i_D + e'_Q i_Q \quad (8.62, 8.63)
\]

The equation of motion is

\[
2H \frac{ds}{dt} = -(T_m - T_c) \quad (8.64)
\]

The differential equations, for $n$ machines, can be linearized and written in matrix form as

\[
\Delta \dot{E} = P \Delta E + X \Delta i + R \Delta s \quad (8.65)
\]

where

\[
P = \begin{bmatrix}
- \frac{1}{T'_{a1}} & \omega_o s_1 \\
- \omega_o s_1 & - \frac{1}{T'_{a1}} \\
- \frac{1}{T'_{a2}} & \omega_o s_2 \\
- \omega_o s_2 & - \frac{1}{T'_{a2}} \\
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
x_1 - x'_1 \\
x_1 - x'_1 \\
x_2 - x'_2 \\
\vdots
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
\omega_o e_Q' \\
- \omega_o e_D' \\
\omega_o e_Q' \\
- \omega_o e_D' \\
\vdots
\end{bmatrix}
\]

\[
\Delta s = \begin{bmatrix}
\Delta s_1 \\
\Delta s_2 \\
\vdots
\end{bmatrix}
\]

The linearized equation of motion, assuming $\Delta T_m = 0$, for $n$ machines, in matrix form, is
where

\[
\Delta s = e \Delta i + i \Delta E
\]  

(8.66)

\[
e = \begin{bmatrix}
e_{D1}' / 2H_1 & e_{Q1}' / 2H_1 \\
e_{D2}' / 2H_2 & e_{Q2}' / 2H_2 \\
\end{bmatrix}, \quad i = \begin{bmatrix}
i_{D1} / 2H_1 & i_{Q1} / 2H_1 \\
i_{D2} / 2H_2 & i_{Q2} / 2H_2 \\
\end{bmatrix}
\]

As before, the voltage equations in linearized form can be written as

\[
\Delta E = Z_M \Delta i + \Delta e
\]  

(8.67)

where

\[
Z_M = \begin{bmatrix}
r_1 - x_1' & x_1' & r_1 & r_2 - x_2' & x_2' & r_2 \\
\end{bmatrix}
\]

In the induction machine equations the axis voltages and currents are referred to a set of synchronously rotating axes which can be assumed to coincide with the network reference axes. Therefore, for the induction machines the transformation matrix \( T \) will be a unit matrix, and the vector \( \Delta \delta \) will be a null vector.

For the complete system (8.28), (8.58) and (8.67) are written as one matrix equation before forming (8.51).

The differential equations for the internal voltages of all the synchronous and induction machines can be collected and written in matrix form as

\[
\Delta \dot{E} = P \Delta E + X \Delta i + R \Delta s + Q \Delta x_{ex}
\]  

(8.68)

The new \( P \) and \( X \) matrices for the mixed machine model will be given by

\[
P = \begin{bmatrix}
P_{\text{synchronous machines}} & \overline{P} \\ 
P_{\text{induction machines}} & \overline{P} \\
\end{bmatrix}, \quad X = \begin{bmatrix}
X_{\text{synchronous machines}} & \overline{X} \\ 
X_{\text{induction machines}} & \overline{X} \\
\end{bmatrix}
\]

The matrix \( R \) is augmented with appropriate number of zeros.

Substituting the expression for \( \Delta i \) from (8.51) and rearranging, (8.68) reduces to

\[
\Delta \dot{E} = C \Delta \delta + D \Delta E + R \Delta s + Q \Delta x_{ex}
\]  

(8.69)

where

\[
C = XA \quad \text{and} \quad D = P + XB
\]
The equations of motion for all the synchronous and induction machines can be written as one matrix equation of the form of equation (8.54) with appropriate modifications to the vectors and matrices. The new $i$ and $e$ matrices will be given by

$$
i = \begin{bmatrix}
\begin{array}{c}
i_{\text{syn. machine model 2}} \\
\text{zero elements for} \\
\text{classical model} \\
i_{\text{induction machine}}
\end{array}
\end{bmatrix}
$$

$$
e = \begin{bmatrix}
\begin{array}{c}
e_{\text{syn. machine model 2}} \\
\text{zero elements for} \\
\text{e}_{\text{syn. machine classical}} \\
e_{\text{induction machine}}
\end{array}
\end{bmatrix}
$$

By eliminating $\Delta i$ the desired equation of the form of (8.55) will be obtained. The coefficient matrix can then be formed as shown in (8.57).

**Singularity of the coefficient matrix**

The coefficient matrix of a multimachine system is singular in the absence of an infinite bus and when absolute values of the machine angles are considered -- the columns corresponding to the $\Delta \delta$'s will add up to zero -- resulting in a zero eigenvalue. This is because the system is actually of order $(m \times n - 1)$ and not $m \times n$, where $m$ is the number of state variables per machine, assuming all the machines are represented similarly. The singularity can be avoided, if desired, by expressing the angles of the machines with reference to a particular machine taken as a reference, thus reducing the number of variables by one. This calls for a few minor changes in the coefficient matrix. For example, taking the $n$th machine as reference, the required changes can be achieved by deleting the row and column corresponding to $\Delta \delta_n$ and placing $-1$ in the intersection of the rows corresponding to $\Delta \delta_i$'s ($i = 1, 2, \ldots, n-1$) and the column corresponding to $\Delta \omega_n$. Apart from eliminating the zero eigenvalue the nonzero eigenvalues will remain unchanged regardless of which machine is taken as reference. This can be demonstrated by applying the fundamental properties of determinants.

When an infinite bus is present and the machine angles are measured with reference to that bus, the system will be of order $m \times n$ and the original coefficient matrix will be non-singular.

**Procedure for Small-Disturbance Stability Studies**

In the state space method of small-disturbance stability analysis, one or more operating conditions whose stability properties need to be investigated would be selected and the state model would be derived following the method(s) discussed earlier. The eigenvalues and eigenvectors would then be computed using a standard eigenvalue computing package. For efficiency, these two steps could be combined into one computer program. Conclusion about stability and system response can then be drawn from an analysis of the eigenvalues and eigenvectors. The program can be readily adapted to evaluate small-disturbance stability limits under given sets of assumptions, impact on small-disturbance performance in response to changes in system and control parameters and network flow, and also to meet various other objectives.
In the state space approach of small-disturbance stability analysis the whole power system is represented by one linear vector matrix equation and hence the eigenvalues contain information on the dynamic response of the whole set of system variables rather than a few in which we are usually interested. Any change in the system dynamic performance due to a change in the system parameter can only be assessed by a proper interpretation of the eigenvalues. A change in the control parameter of a particular machine may have effects on other machines which may be quite remote and this will be reflected in the eigenvalues. The method is thus invaluable in identifying the unstable modes not apparent in time domain simulations, their sources and possible fixes of the problem.

From the solution of the vector differential equation $\mathbf{x} = \mathbf{Ax}$, as given in Chapter 2, it is clear that for a stable system a real eigenvalue will give rise to an exponentially decaying non-oscillatory term, the decay time constant being the reciprocal of the eigenvalue, whereas a pair of complex conjugate eigenvalues will give rise to an oscillatory term having an angular frequency equal to the imaginary part, the oscillation decaying exponentially with a time constant which is the reciprocal of the real part. Thus in a linear dynamic system the eigenvalues correspond to the natural modes of response. An eigenvalue with a small negative real part will indicate the presence of a transient term which is slow to die out. However, the relative magnitude of that term in a particular variable will depend on the particular element of the associated eigenvector.

In multi-machine systems, in general, some of the eigenvalues are more sensitive to variations of control parameters and operating conditions than others. Since in the output the eigenvalues are printed in descending order of magnitude, it is sometimes difficult to follow the locus of any particular eigenvalue with these changes. However, the sensitivity of a particular eigenvalue with respect to a parameter may be calculated, if desired.

**Eigen-sensitivity Analysis**

Eigenvalue sensitivity information can be useful in determining the impact of changes of system parameters and control on system modes and, therefore, parameters that have the most influence on system damping can be identified. The expressions of sensitivities of eigenvalue $\lambda_i$ and eigenvector $\mathbf{x}_i$ of the coefficient matrix $\mathbf{A}$ with respect to a system parameter $\alpha$ have been derived in Appendix A, and are given by

$$\frac{\partial \lambda_i}{\partial \alpha} = \frac{y_i \frac{\partial A}{\partial \alpha} x_i}{y_i x_i}$$

and

$$\frac{\partial x_i}{\partial \alpha} = \sum_{j=1}^{n} \mu_{ij} x_j,$$

respectively, where

$$\mu_{ij} = \frac{y_j \frac{\partial A}{\partial \alpha} x_i}{(\lambda_i - \lambda_j) y_i' x_j}$$

and $y_i$ is the $i$th eigenvector of the transpose of $\mathbf{A}$. 
An important advantage of the method of deriving the state model as described previously is that, in the coefficient matrix the elements involving any particular system parameter can be easily located. The partial derivatives of the coefficient matrix with respect to a system parameter as required in the calculation of eigen-sensitivities can therefore be readily formed. For example, if the sensitivity of the eigenvalue $\lambda_i$ with respect to a parameter of the excitation control system, say the excitation gain constant of the $i$th machine, $K_{Ai}$, is desired, it is easily seen that only the matrices $K_{ex}$, $G$ and $H$ contain this parameter. The partial derivatives with respect to $K_{Ai}$ of all the other submatrices forming the final $A$ matrix will be null matrices. The derivative of $K_{ex}$ can be easily formed by inspection. The derivatives of $G$ and $H$ for the machine model 2 can be seen to be

$$\frac{\partial G}{\partial K_{Ai}} = \frac{\partial L}{\partial K_{Ai}} \left[ Y_N + T^{-1}Z_M^{-1}T \right]^{-1} T^{-1}Z_M^{-1}$$

and

$$\frac{\partial H}{\partial K_{Ai}} = - \frac{\partial L}{\partial K_{Ai}} \left[ Y_N + T^{-1}Z_M^{-1}T \right]^{-1} T^{-1} \left[ M + TY_N N \right]$$

Since of the submatrices of which $G$ and $H$ are formed only $L$ contains $K_{Ai}$.

**System Performance**

One advantage of formulating the power system small-disturbance problem in terms of state variables is that many of the results from control theory can be applied. The general form of the state space model consists of the pair of equations (see Chapter 2)

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (8.70)

$$y = Cx + Du$$  \hspace{1cm} (8.71)

Note: In the discussion of small-disturbance stability through a state space approach presented earlier in the chapter, the second term of the right hand side of equation (8.70) was absent. This was due to the fact that the control vector was either constant or expressed beforehand in terms of the state variables in which case the two terms on the right hand side of equation (8.70) were combined together.

If the system is unstable or the eigenvalues are such that satisfactory performance is not expected, the system can be stabilized and the performance improved by employing feedback. The input $u(t)$ can be chosen to be function of the state variables, so that $u = -hx$, where $h$ is the feedback matrix. The state equation (8.70) then becomes

$$\dot{x} = Ax - Bhx = [A - Bh]x = A'x$$  \hspace{1cm} (8.72)

By appropriate selection of the elements of $h$, the system can be stabilized and performance improved.

Most of the results available from control theory are, however, impractical for large systems such as a realistically sized power system. These are therefore of academic interest only, although interesting applications are possible in small systems. Two of these are briefly discussed here.

The first one uses the concepts of optimal control. The linearized equations, after dropping the prefix $\Delta$, are written in the form of (8.70). A quadratic performance index of the form
SMALL DISTURBANCE STABILITY

\[ J = \int_0^T (x'Qx + u'Ru)dt \]  \hspace{1cm} (8.73)

is then chosen and minimized. The elements of the positive definite matrices \( Q \) and \( R \) are selected after appropriate weighting which is usually guided by experience. The object of minimizing \( J \) is to minimize the sum of the error response and the control effort. It may be noted that taking \( Q \) as a unit matrix, \( \int_0^T x'Qx dt \) constitutes the familiar integral squared error criterion.

The methods as followed in deriving state models of the form of (8.3) can be used in deriving equation (8.70) from a set of differential and algebraic equations. For example, equation (8.2) will first be expressed as

\[
\begin{pmatrix}
\dot{x} \\
0
\end{pmatrix}
= \begin{pmatrix}
A_1 & A_2 & A_3 \\
\end{pmatrix}
\begin{pmatrix}
 x \\
 y
\end{pmatrix}
+ \begin{pmatrix}
 B_1 \\
 B_2
\end{pmatrix} u
\]  \hspace{1cm} (8.74)

where \( u \) is the control vector.

From (8.74) (8.70) follows where

\[
A = A_1 - A_2 A_4^{-1} A_3 \quad \text{and} \quad B = B_1 - A_2 A_4^{-1} B_2
\]

It is shown in standard texts on control theory that for optimal control (minimum \( J \)) \( u \) is given by

\[ u = - R^{-1} B' K x \]  \hspace{1cm} (8.75)

where \( K \) is a positive definite symmetric matrix satisfying the Riccati equation

\[ - A'K - KA + KBR^{-1}B'K - Q = 0 \]  \hspace{1cm} (8.76)

Therefore the system (8.70) with optimal control reduces to

\[ \dot{x} = A^1 x \]

where

\[ A^1 = A - BR^{-1} B'K \]

Although the method is attractive it has to be limited to low order systems because of the necessity of solving equations (8.76) to obtain the matrix \( K \).

Also, it can be seen from equation (8.75) that the optimal control is given by a linear combination of all the state variables not all of which are directly measurable and available and thus the real advantage of a rigorous theory may not be realized. This is so in a multi-machine system where the optimal control law will suggest feedbacks of variables of one machine into another. These, along with a less important fact that the optimal control law depends on the choice of the matrices \( Q \) and \( R \) in the expression for the performance index, do not render the optimal control theory particularly useful in the design of the controls in a multi-machine system in which power system engineers are more interested. As it stands it can be helpful only in the design of controls in the relatively simple single machine-infinite bus systems.

A useful alternative, which is more suitable for small multi-machine systems, will be not to establish the optimal control law in the way as described above, but to preselect the controls as combinations of some of the measurable and available state variables which are likely to produce
improved dynamic response, then obtain an estimate of \( \int_0^\infty x'Qx\, dt \) through the use of Liapunov function and adjust the control parameters so as to reduce the numerical value of the integral. Thus once again we start with equation \( \dot{x} = Ax \). To calculate the performance index given by

\[
J = \int_0^\infty x'Qx\, dt
\]  

(8.77)

where \( Q \) is a positive definite or semidefinite symmetric matrix, we make use of Liapunov’s theorem on linear time-invariant systems.

Choosing a Liapunov function and its time derivative as

\[
V(x) = x'Px
\]

(8.78)

\[
\dot{V}(x) = -x'Qx
\]

(8.79)

where the positive definite symmetric matrix \( P \) is determined from the equation

\[
A'P + PA = -Q
\]

(8.80)

we have

\[
\int_0^\infty x'Qx\, dt = -\int_0^\infty dV(x)
\]

For an asymptotically stable system \( x(\infty) = 0 \), and noting that \( V(0) = 0 \), we have

\[
\int_0^\infty x'Qx\, dt = V(x_o) = x_o'P\,x_o
\]

(8.81)

Thus the performance index is equal to the value of the Liapunov function \( x'Px \) at \( t = 0 \).

**Solution of the equation** \( A'P + PA = -Q \)

Equation (8.80) has a unique solution for \( P \) if \( \lambda_i + \lambda_j \neq 0 \) for all \( i, j \), where \( \lambda_i \) are the eigenvalues of the coefficient matrix \( A \). This condition will obviously be satisfied for an asymptotically stable system. It is assumed that before trying to optimize a system performance its stability has been properly checked.

To solve for the matrix \( P \) we need to find the elements on and above (or below) the leading diagonal of the matrix, which are given by the solution of the \( n(n+1)/2 \) equations arising out of the matrix equation (8.80). Although a direct solution is straightforward, the computation time goes up considerably with the size of the system. For anything but the very small systems a direct solution is not economical.

The computational burden can be reduced somewhat by first diagonalizing the matrix \( A \) using the relationship \( M^{-1}AM = \Lambda \) (see Appendix A), where \( \Lambda \) is a diagonal matrix composed of the eigenvalues of \( A \) (assuming the eigenvalues are distinct) and \( M \) is a square matrix whose columns are the corresponding eigenvectors. Thus, we have

\[
A = M\Lambda M^{-1} \quad \text{and} \quad A' = [M^{-1}]'\Lambda M'
\]

Using these relationships (8.80) reduces to

\[
[M^{-1}]'\Lambda M'P + PM\Lambda M^{-1} = -Q
\]
Pre-multiplying and post-multiplying both sides by $M'$ and $M$, respectively, the above reduces to

$$\Delta V + VA = -S$$

(8.82)

where

$$V = M'PM \quad \text{and} \quad S = M'QM$$

Equation (8.82) can be written in expanded form as

$$
\begin{bmatrix}
2\lambda_1 v_{i1} & (\lambda_1 + \lambda_2)v_{i2} & (\lambda_1 + \lambda_3)v_{i3} & \cdots \\
(\lambda_1 + \lambda_2)v_{i2} & 2\lambda_2 v_{i2} & (\lambda_2 + \lambda_3)v_{i2} & \cdots \\
(\lambda_1 + \lambda_3)v_{i3} & (\lambda_2 + \lambda_3)v_{i3} & 2\lambda_3 v_{i3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
= 
\begin{bmatrix}
s_{i1} & s_{i1} & s_{i1} & \cdots \\
s_{i2} & s_{i2} & s_{i2} & \cdots \\
s_{i3} & s_{i3} & s_{i3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

The elements of $V$ can thus be easily calculated by equating the corresponding elements. Once $V$ is found, $P$ is calculated from

$$P = [M^{-1}]'VM^{-1}$$

Thus the performance index for various combinations of parameter values can be computed and a selection of the parameter values for acceptable performance can be made. If desired, the performance index can be minimized using a suitable function minimization technique.

Note that in the expression for the performance index (8.77) not all the state variables need to be included. Also, the procedure can be readily adapted when the variables in the performance index are chosen to be output (non-state) variables. For example, if we wish to estimate or minimize

$$\int_0^\infty \sum_{i=1}^n \Delta V_i^2 \, dt,$$

we note that from (8.17) $\Delta V_i = V \Delta e$ and from (8.50) (assuming machine model 2) $\Delta e = C \Delta x$. Therefore the performance index becomes

$$\int_0^\infty \sum_{i=1}^n \Delta V_i^2 \, dt = \int_0^\infty [\Delta V_i] [\Delta V_i] \, dt = \int_0^\infty \Delta x' C' V' V C \Delta x \, dt = \int_0^\infty \Delta x' Q \Delta x \, dt$$

which is of the same form as (8.77).

References

CHAPTER 9
TURBINE-GENERATOR SHAFT TORSIONALS

Most system disturbances, ranging from transmission line switching to electrical faults and subsequent circuit breaker operations, produce transient oscillatory torques as well as step changes in the torque level at the generator rotor air gap. These torques, whose magnitudes can sometimes be several times the steady state level under full load condition, induce torsional response in the turbine-generator shaft system. The resulting oscillations in the shaft system can subject it to strain-cycles causing a loss of fatigue life. Since the shaft system generally has a very low damping, oscillations once started can continue for a long time. At relatively high oscillation amplitudes fatigue life expenditure can be considerable.

Under abnormal operating conditions the amount of stress the shaft system is subjected to would depend on the nature of the disturbance, network parameters, system configuration, etc. Turbine-generators are generally designed to withstand the effects of short circuits as required by industry standards. Adequate margins are provided for carrying the steady state shaft torque at full load.

The highly oscillatory components of the transient electrical torque usually die out rapidly. Occasionally, however, for example, when there is series capacitor in the transmission line, these torques can be negatively damped due to self-excitation. There can also be resonance, when the oscillation frequency coincides with one of the shaft system natural frequencies. Under these conditions the shaft torque can build up and serious damage to the shaft system may result.

The generator electrical torque or the output power calculated in a conventional large scale stability program does not contain the high frequency component. Computation of the high frequency components requires a detailed machine representation which makes the computation extremely time consuming. Also, the effect of the oscillations on the overall movement of the rotor with respect to the system is not usually appreciable. For this reason, for conventional stability calculations, it is adequate to employ a machine model (for example, one of the models discussed in Chapter 5) that provides only the average torque. As an example, the electrical torque or power following a three-phase terminal short circuit at no load, that would be obtained from one of the synchronous machine models normally used in a conventional stability program, is zero. The actual torque at the instant of short circuit is given by

\[ T_e \approx \frac{V^2}{x^\prime}_d \sin \omega_o t \quad (9.1) \]

The above expression is arrived at by neglecting all resistances (stator and rotor), and assuming \( x^\prime_d \approx x^\prime_q \).

The fundamental frequency torque is caused by the dc offset of the transient current. It decays rapidly -- approximately with the armature time constant. However, its initial value can be several times the normal full load torque. The effect of stator and rotor resistances is to modify the above expression somewhat and add additional terms which form the unidirectional components. All these components decay at various stator and rotor time constants. For a detailed discussion of the short circuit torques see Reference 3.
Electrical torques generated due to other disturbances and switching operations are of a similar nature. Under certain conditions the torques generated are in excess of that for a three-phase terminal short circuit.

**Problem**

Show that the electrical torque due to synchronizing an unloaded generator, out of phase, against an infinite bus, neglecting all resistances, subtransient saliency, voltage regulator and governor actions, and assuming constant rotor speed and unity terminal and infinite bus voltage, is given by

\[
T_e \approx \frac{\sin \delta}{x_d^t + x_c} - \frac{2 \sin \frac{\delta}{2} \cos (\omega_o t + \frac{\delta}{2})}{x_d^s + x_c},
\]

where

- \( \delta \) = angle by which the generator rotor leads the infinite bus
- \( x_d^t \) = generator direct axis subtransient reactance
- \( x_c \) = reactance between generator terminal and infinite bus.

Note that the maximum value of the torque occurs when \( \delta = 120^\circ \).

**A Basic Analysis of the Shaft Torque Problem**

In order to understand the effect of electrical system disturbances on the turbine-generator shaft system consider the simple example shown in Figure 9.1

![Fig.9.1 A simple turbine-generator torsional system.](image)

In the absence of any interaction between the mechanical and the electrical system, the effect of an electrical system disturbance can be analyzed by applying the change in the electrical torque as a driving function to the equations of the mechanical system. The effect of interaction between the mechanical and electrical system will be considered later.

The equations of motion of the turbine-generator shaft system can be written as

\[
\frac{2H_1}{\omega_o} \frac{d^2 \theta_1}{dt^2} + D_{12} (\dot{\theta}_1 - \dot{\theta}_2) + D_1 \ddot{\theta}_1 + K_{12} (\theta_1 - \theta_2) = T_m
\]

\[
\frac{2H_2}{\omega_o} \frac{d^2 \theta_2}{dt^2} + D_{12} (\dot{\theta}_2 - \dot{\theta}_1) + D_2 \ddot{\theta}_2 + T_e = K_{12} (\theta_1 - \theta_2)
\]

where \( H_1 \) and \( H_2 \) are the inertias of the turbine and the generator rotor, respectively, \( T_m \) is the applied mechanical torque, \( T_e \) is the generator electrical torque, \( K_{12} \) is the torsional stiffness of the shaft, \( D_1 \) and \( D_2 \) are the damping coefficients applicable to the turbine and the generator.
rotor, and \( D_{12} \) is the damping coefficient to account for the damping arising within the shaft material. \( \omega_b = 2\pi f_o \), where \( f_o \) is the electrical system frequency (60 Hz).

Assuming a constant applied mechanical torque, and \( H_1 D_2 = H_2 D_1 \), for simplicity, (9.3) and (9.4) can be linearized and combined as

\[
\frac{2H_1 H_2}{\omega_o (H_1 + H_2)} \frac{d^2}{dt^2} \Delta \theta_{12} + D_{12} \frac{d}{dt} \Delta \theta_{12} + K_{12} \Delta \theta_{12} = \frac{H_1}{H_1 + H_2} \Delta T_e
\]  

(9.5)

Assuming the disturbance torque \( \Delta T_e \) to be of the form \( T e^{-\alpha t} \cos(\omega t + \phi) \), equation (9.5) can be written as, writing \( \theta \) for \( \Delta \theta_{12} \) for brevity,

\[
\frac{d^2 \theta}{dt^2} + 2\sigma \frac{d\theta}{dt} + \omega_n^2 \theta = A e^{-\alpha t} \cos(\omega t + \phi)
\]  

(9.6)

where

\[
\sigma = \frac{D_{12} \omega_o (H_1 + H_2)}{4H_1 H_2}, \quad \omega_n^2 = \frac{K_{12} \omega_o (H_1 + H_2)}{2H_1 H_2}
\]

and

\[
A = \frac{T \omega_o}{2H_2}
\]

\( \omega_n \) corresponds to the natural frequency of the turbine-generator shaft system.

The general solution to equation (9.6) is of the form

\[
\theta = \left[ C_1 \cos \sqrt{\omega_n^2 - \sigma^2} t + C_2 \sin \sqrt{\omega_n^2 - \sigma^2} t \right] e^{-\alpha t} + \frac{A}{R} e^{-\alpha t} \cos(\omega t + \phi + \beta)
\]  

(9.7)

where

\[
R = \sqrt{(\omega_n^2 - \omega^2 + \alpha^2 - 2\sigma \alpha)^2 + 4\omega^2 (\alpha - \sigma)^2}
\]  

(9.8)

and

\[
\beta = \tan^{-1} \frac{2\omega (\alpha - \sigma)}{\omega_n^2 - \omega^2 + \alpha^2 - 2\sigma \alpha}
\]  

(9.9)

\( C_1 \) and \( C_2 \) are the two constants to be determined from the given initial conditions.

For example, for given initial values \( \theta_o \) and \( \dot{\theta}_o \), \( C_1 \) and \( C_2 \) are calculated as

\[
C_1 = \theta_o - A \omega \cos(\phi + \beta)
\]

\[
C_2 = \frac{1}{\sqrt{\omega_n^2 - \sigma^2}} \left[ \dot{\theta}_o + \sigma \theta_o - A [ (\sigma - \alpha) \cos(\phi + \beta) - \omega \sin(\phi + \beta) ] \right]
\]

Therefore the complete solution can be written as
Note that the torque that the shaft will be subjected to is obtained by multiplying \( \theta \) by the spring constant \( K_{12} \).

Equation (9.10) tells a great deal about the amount and type of stress the shaft system will be subjected to under various electrical system disturbances.

The first two terms represent the free motion at shaft system natural frequency (\( \approx \omega_n \)). These die out at a rate determined by \( \sigma \) which is usually very small. The first term is due to the finite initial values of \( \theta \) and \( \dot{\theta} \). For zero initial values, i.e., starting with no initial perturbation, this term will be absent. Note that, depending on the initial perturbation, this term can greatly influence the shaft movement and hence the stress level.

The third term represents the forced vibration due to the applied torque. It is at the same frequency as that of the applied torque but displaced in phase. The initial magnitude of this (as well as the second term) is dependent on the amplitude of the applied torque and the magnitude of \( R \) given by (9.8).

Note that for a given shaft system the magnitude of \( R \) depends largely on the frequency of the applied torque. Depending on the nature of the disturbance, the applied torque may be composed of one or more unidirectional and/or oscillatory components. For example, in a unit trip from full load (full load rejection), the air-gap torque suddenly changes from full load value, \( T_eo \), to zero. Therefore, initially \( \Delta T_e = -T_eo \). In the case of other disturbances, \( \Delta T_e \), in general, will have both unidirectional and oscillatory components. The frequency of oscillation can range from inter-machine swing frequency (1/2 - 2 Hz) to power frequency (60 Hz) and its second harmonic. All these components generally decay rapidly with various time constants. Occasionally, however, the decay rate can be very small and even negative.

Assuming \( \omega_n \) to be in the 20 - 40 Hz range and assuming nominal decay rates for the applied torques, approximate values of \( R \) corresponding to the frequencies of interest can be estimated from equation (9.8). It can be seen that \( R \approx \omega^2 \) when the applied torque is at power frequency or higher. \( R \approx \omega_n^2 \) when the applied torque is unidirectional or at a low frequency, such as the inter-machine swing frequency. When the frequency of the applied torque is close to \( \omega_n \), \( R \approx 2 \omega_n \alpha \).

Since \( 2 \omega_n \alpha \ll \omega_n^2 \ll \omega_e^2 \), we can conclude that the impact of an applied torque of a given amplitude is greatest when its frequency is close to the shaft natural frequency (forming resonance). Also, for low frequency or unidirectional applied torques, the impact is considerably worse than that for torques at power frequency or higher.
For most network disturbances $\alpha$ is large (which corresponds to short time-constants) and therefore the third term of equation (9.10) decays very rapidly, although the initial peak can be high if the applied torque has a high enough initial amplitude. However, the impact of the motion imparted to the shaft system by the applied torque, as depicted by the second term of equation (9.10), can be considerable, since it decays at a very slow rate due to the small value of $\sigma$.

The significance of the first term of equation (9.10) lies in fault clearing and high-speed reclosing sometimes practiced as a means of improving stability limit. Any switching operation is accompanied by transient oscillatory torque at the generator rotor air-gap. A system fault will set the shaft system in motion, given by equation (9.10). If the system was in a steady state before the fault, the first term in equation (9.10) will be absent. If the fault is cleared too soon, before the forced vibration had sufficient time to decay, the initial perturbation for the transients generated by fault clearance can be considerable. As a result the shaft system will be subjected to a much higher stress level. The effect can be cumulative depending on the phase relationships of the various terms, as can be seen from equation (9.10).

From the point of view of synchronous stability a fault clearing time faster than 3 cycles is hardly worthwhile. During the first cycle or two, the unidirectional components of the electrical torque due to the stator and rotor losses caused by the transient induced currents can be substantial. The net accelerating torque immediately following a fault is therefore very small. It can even be negative, causing the rotor to retard momentarily. Due to the modeling restriction, the unidirectional component of the electrical torque is not fully accounted for in a conventional large scale stability program. During the first three cycles into the fault the actual forward movement of the rotor will therefore be very little. A faster clearing is therefore not likely to improve system stability perceptively. However, as explained above, it can increase the stress on the shaft system considerably.

A delayed clearing can also result in increased shaft torque and therefore should be avoided, even if it is acceptable from system stability point of view. This is because, with delayed clearing, the generator rotor would have advanced substantially by the time the fault is cleared. The transient torque following fault clearing will therefore have a slowly varying component with relatively high amplitude, and this will impart considerable shaft movement.

From equation (9.10), it can be seen that the actual torsional movement of the shaft depends on various factors. At any instant of time, the deviation from the steady state position is determined by the shaft system natural frequency, the frequency of the applied torque, various phase relationships and damping. Most of these quantities will vary from one machine to another. Since the shaft position and speed at the instant of fault clearing will serve as the initial conditions for the transients following the fault clearing, the severity of these transients and the resulting shaft stress would depend on the point on the transient wave at which the fault clearance takes place. Since the actual wave shapes would be different for different machines, and cannot be predicted exactly in advance, the best instant of fault clearing from shaft torsional point of view cannot be determined. All things considered, the present practice of 3 - 3.5 cycles clearing appears to be the most desirable.

Equation (9.10) also suggests the possibility of the shaft system being subjected to high stresses following a fast reclosure. The third term in the equation will probably disappear by the time reclosure takes place. However, the first two terms may still be significant due to the very low damping within the shaft system. This, combined with the fact that there may be considerable
angle difference between the generator rotor and the electrical system at the instant of reclosure (c.f. the problem on synchronizing out of phase), could have the potential of causing even greater stress on the shaft system. It may therefore be advisable to avoid fast reclosure unless it is really necessary for maintaining stability. In those rare situations where fast reclosure is required, a thorough analysis of the shaft system stresses should be performed.

Note that the possibility of closing into a fault (unsuccessful reclosure) presents a special hazard in the practice of fast reclosure, especially if the fault is a three-phase short circuit close to the machine terminal, since the transients following such unsuccessful reclosure and subsequent clearing will be much more severe than normally encountered.

A great majority of system faults involve only one phase. Single phase clearing and reclosing in such situations will not only improve system stability but also reduce shaft stresses considerably. The cost of providing effective compensation for arc suppression, where required, has been shown to be moderate. Therefore, a preferred alternative to three phase clearing and high speed reclosing would be single pole switching. On those rare occasions when the fault is a close-in three phase short circuit, fast reclosing can be blocked and alternative means of maintaining system integrity can be resorted to.

Under certain circumstances the applied torque may be lightly damped or the damping may even be negative, as a result of self-excitation due to the presence of series capacitors in the system. The applied torque and consequently the shaft motion will then build up, as seen from equation (9.10), when $\alpha$ is negative. Self-excitation will be considered in more detail later. Even when the damping is positive, it could be small and the transient torque developed following a system disturbance could have a frequency close to the torsional natural frequency of the shaft system. The amplitude of the forced vibration can then reach a very high level, since $R \to 0$ at $\omega = \omega_n$ for small values of $\alpha$ and $\sigma$ as seen from equation (9.8). This is the familiar phenomenon of resonance in forced vibration.

There can also be interaction between the shaft torsional system and the electrical system in the presence of series capacitors, which could cause instability under conditions of resonance or near resonance. Interaction between the shaft system and certain excitation and governor control systems through the electrical system is also possible. This occurs when the speed signal utilized in the control (in the case of excitation control, speed signal is used for power system stabilization purposes) is derived at a shaft location which contains significant levels of torsional natural frequency components. A method of simultaneously analyzing the effect of all these interactions will be discussed later.

In the discussion of the torsional oscillation of the simple system shown in Figure 9.1, an equation describing the relative motion of the turbine with respect to the generator rotor was developed and solved. All the twisting was assumed to take place in the shaft. For analyses of shaft stresses and torsional interactions with the electrical system this is permissible. The relative motion of the turbine with respect to the generator, i.e., the twisting of the shaft is referred to as the torsional mode. In conventional stability study the motion of the complete turbine-generator system with respect to the electrical system (or other generators) is of interest. This is the electrical system mode. Since the relative movement of the turbine with respect to the generator rotor is small compared with the movement of the rotor with respect to the system, the torsional mode can be neglected and the turbine-generator shaft system can be treated as a single rotating mass in conventional stability studies.
TURBINE-GENERATOR SHAFT TORSIONALS

In the system of Figure 9.1 there is only one torsional mode. Generally, there will be more than one turbine element, and there may also be a rotating exciter. The number of torsional modes in such cases will be one less than the number of rotating elements.

**Modes and Mode Shapes in a Turbine-Generator Shaft Torsional System**

In the simple example considered so far there was only one torsional mode (or degree of freedom), since there were only two rotating masses connected by a shaft. In a practical system there will usually be more than one turbine section (such as high pressure, intermediate pressure, low pressure, etc.), as well as the exciter connected to the generator rotor by a shaft. For analyses of shaft torsional stresses at or below synchronous frequency (60 Hz), the turbine-generator system can be represented by a spring-mass system, similar to that in the simple system, by treating the various elements as point masses, assuming negligible twisting within the elements. For example, a turbine generator with three turbine elements and an exciter can be represented as in Figure 9.2.

![Fig. 9.2 A turbine-generator shaft torsional model.](image)

The equations of motion of the above system can be written, similar to those of the simple system, as

\[
\frac{2H_1}{\omega_0} \frac{d^2 \theta_1}{dt^2} + D_{12} \frac{d}{dt} (\theta_1 - \theta_2) + D_{11} \frac{d \theta_1}{dt} + K_{12} (\theta_1 - \theta_2) = T_{ml}
\]  
\(9.11\)

\[
\frac{2H_2}{\omega_0} \frac{d^2 \theta_2}{dt^2} + D_{12} \frac{d}{dt} (\theta_2 - \theta_1) + D_{22} \frac{d \theta_2}{dt} + D_{23} \frac{d \theta_3}{dt} + K_{12} (\theta_2 - \theta_1) + K_{23} (\theta_2 - \theta_3) = T_{m2}
\]  
\(9.12\)

\[
\frac{2H_3}{\omega_0} \frac{d^2 \theta_3}{dt^2} + D_{23} \frac{d}{dt} (\theta_3 - \theta_2) + D_{33} \frac{d \theta_3}{dt} + D_{34} \frac{d \theta_4}{dt} + K_{23} (\theta_3 - \theta_2) + K_{34} (\theta_3 - \theta_4) = T_{m3}
\]  
\(9.13\)

\[
\frac{2H_4}{\omega_0} \frac{d^2 \theta_4}{dt^2} + D_{34} \frac{d}{dt} (\theta_4 - \theta_3) + D_{44} \frac{d \theta_4}{dt} + D_{45} \frac{d \theta_5}{dt} + K_{34} (\theta_4 - \theta_3) + K_{45} (\theta_4 - \theta_5) = -T_e
\]  
\(9.14\)

\[
\frac{2H_5}{\omega_0} \frac{d^2 \theta_5}{dt^2} + D_{45} \frac{d}{dt} (\theta_5 - \theta_4) + D_{55} \frac{d \theta_5}{dt} + K_{45} (\theta_5 - \theta_4) = -T_{ex}
\]  
\(9.15\)

Since the shaft spring constants, \(K_{12}\), etc., are large, \((\theta_1 - \theta_2)\), etc. are small and can be assumed to be zero, i.e., \(\theta_1 \approx \theta_2 \approx \theta_3 \cdots\), when the motion of the whole turbine-generator shaft system with respect to the electrical system or other generators is of interest, as in a conventional stability study. With \(\theta_1 \approx \theta_2 \approx \theta_3 \cdots\), the above equations can be combined into
\[
\frac{2H}{\omega_o} \frac{d^2 \delta}{dt^2} + D \frac{d\delta}{dt} + T_e = T_m
\]  
(9.16)

where

\[
H = H_1 + H_2 + \cdots + H_5 \\
T_m = T_{m1} + T_{m2} + \cdots \\
D = D_{11} + D_{22} + \cdots
\]

\(\delta \approx \theta_1 \approx \theta_2 \approx \cdots\) is the angle with respect to the electrical system. \(T_e\) is small and can either be neglected or included with \(T_m\).

Equation (9.16) is the familiar swing equation used in conventional stability studies.

Assuming constant mechanical torque and \(T_e\), equations (9.11) through (9.15) can be linearized and written in matrix form as

\[
\mathbf{H} \frac{d^2 \Delta \mathbf{\theta}}{dt^2} + \mathbf{D} \frac{d\Delta \mathbf{\theta}}{dt} + \mathbf{K} \Delta \mathbf{\theta} = \Delta \mathbf{T}
\]  
(9.17)

where

\[
\mathbf{H} = \text{diag} \left[ \frac{2H_1}{\omega_o}, \frac{2H_2}{\omega_o}, \ldots, \ldots \right]
\]

\[
\mathbf{D} = \begin{bmatrix}
D_{11} + D_{12} & -D_{12} & & & \\
-D_{12} & D_{12} + D_{22} + D_{23} & -D_{23} & & \\
& -D_{23} & D_{23} + D_{33} + D_{34} & -D_{34} & \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots
\end{bmatrix}
\]

\[
\mathbf{K} = \begin{bmatrix}
K_{12} & -K_{12} & & & \\
-K_{12} & K_{12} + K_{23} & -K_{23} & & \\
& -K_{23} & K_{23} + K_{34} & -K_{34} & \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots
\end{bmatrix}
\]

\[\Delta \mathbf{\theta} = [\Delta \theta_1 \ \Delta \theta_2 \ \cdots \ \Delta \theta_5] \quad \text{and} \quad \Delta \mathbf{T} = [0 \ 0 \ \Delta T_e \ 0]^T\]

\(\Delta T_e\) could be positive or negative; here it is assumed positive.

For an analysis of the torsional modes the damping terms can be neglected, since these are small. Equation (9.17) can then be rearranged as

\[
\frac{d^2 \mathbf{x}}{dt^2} = \mathbf{A} \mathbf{x} + \mathbf{T}
\]  
(9.18)

where
The eigenvalues of the matrix $A$ corresponds to the various modes of oscillation of the system (for a more detailed discussion of modes and mode shapes see Chapter 2). Note that, since the electrical torque has been represented as an independent forcing function, the electrical mode is absent in the above model. Since there are only four independent variables corresponding to the four torsional modes, one of the eigenvalues of the matrix $A$ is zero.

Using a transformation $x = My$, where $M$ is a matrix composed of the eigenvectors of $A$, called the modal matrix, equation (9.18) transforms into

$$\frac{d^2y}{dt^2} = Wy + F \quad (9.19)$$

where

$$W = \text{diag}[-\omega_1^2, -\omega_2^2, \cdots, -\omega_5^2]$$

and

$$F = M^{-1}T$$

$-\omega_1^2, -\omega_2^2, \cdots$ are the eigenvalues of $A$ and $\omega_1, \omega_2, \cdots$ are the torsional natural (angular) frequencies. In the above example there are four non-zero values of $\omega$ corresponding to the four torsional natural frequencies.

Equation (9.19) is the modal equation of the shaft system. In this model the variables are separated from each other and the individual modes can be analyzed independently, as in the simple two mass system considered earlier. The actual shaft motion is obtained by combining the separate responses using the transformation $x = My$. Note that the initial values of the elements of $y$ are obtained from $y_o = M^{-1}x_o$.

**Development of modal equivalents**

For the five mass system of Figure 9.2, equation (9.19) can be broken down as

$$\frac{d^2y_i}{dt^2} = -\omega_i^2y_i + (M^{-1})_{i4} \frac{\omega_o}{2H_4} \Delta T_e \quad i = 1, 2, \cdots, 5 \quad (9.20)$$

where $(M^{-1})_{i4}$ is the element of the $i$th row and 4th column of the inverse of the modal matrix $M$. For a sinusoidal input, $\Delta T_e$, of frequency $\omega$, the response of the system of equations (9.20) is given by
The actual response is obtained from \( x = My \). Note that as the frequency of the sinusoidal input, \( \omega \), approaches one of the natural frequencies, say \( \omega_k \), the particular element \( y_k \) of the vector \( y \) approaches infinity, and it will dominate over other elements. Therefore, at an input frequency close to \( \omega_k \), the system response can be written as

\[
x_i^k = m_{ik} \, y_i^k = \frac{m_{ik}}{\omega_k^2 - \omega^2} \frac{\omega_o}{2H_4} \Delta T_e \quad i = 1, 2, \cdots, 5
\]

(9.22)

The superscript \( k \) denotes the response due to an input of frequency close to the natural angular frequency \( \omega_k \).

Equation (9.22) is also the response of the system given by

\[
\frac{2H_{mi}}{\omega_o} \frac{d^2 x_{mi}}{dt^2} + K_{mi} \, x_{mi} = \Delta T_e
\]

(9.23)

where

\[
H_{mi} = \frac{H_4}{m_{ik} (M^{-1})_{kk}}, \quad K_{mi} = \frac{2H_{mi}}{\omega_o} \omega_k^2
\]

The system represented by equation (9.23) is the modal equivalent of the \( i \)th mass of the shaft torsional system. For an excitation \( \Delta T_e \) at a frequency close to \( \omega_k \), the modal equivalent has the same response as the corresponding mass in the actual shaft torsional system.

As an example, consider the two mass systems shown in Figure 9.1. The linearized equation of motion in matrix form, neglecting damping, is

\[
\frac{d^2}{dt^2} \begin{bmatrix} \Delta \theta_1 \\ \Delta \theta_2 \end{bmatrix} = \begin{bmatrix} -K_{12} \frac{\omega_o}{2H_1} & \frac{\omega_o}{2H_1} \\ \frac{\omega_o}{2H_2} & -K_{12} \frac{\omega_o}{2H_2} \end{bmatrix} \begin{bmatrix} \Delta \theta_1 \\ \Delta \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\omega_o}{2H_2} \end{bmatrix} \Delta T_e
\]

(9.24)

The eigenvalues of the system are

\[
\lambda_1 = 0, \quad \lambda_2 = -K_{12} \frac{H_1 + H_2}{2H_1 H_2}
\]

The corresponding eigenvectors are

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{H_1}{H_2} \\ 1 \end{bmatrix}
\]

Therefore

\[
M = \begin{bmatrix} 1 & -\frac{H_1}{H_2} \\ 1 & \frac{H_1}{H_2} \end{bmatrix} \quad \text{and} \quad M^{-1} = \frac{1}{H_1 + H_2} \begin{bmatrix} H_1 & H_2 \\ H_2 & -H_2 \end{bmatrix}
\]

Note that the system has one non-zero eigenvalue and therefore one natural frequency, given by
\[ \omega_2 = \sqrt{\frac{K_{12}}{\omega_0} \frac{H_1 + H_2}{2H_1H_2}} \]  

(9.25)

Therefore, the modal equivalent of the turbine mass is

\[ \frac{2H_{m1}}{\omega_0} \frac{d^2\theta_{m1}}{dt^2} + K_{m1}\theta_{m1} = \Delta T_e \]  

(9.26)

where

\[ H_{m1} = \frac{H_2}{H_2} = -(H_1 + H_2) \]

\[ = \frac{H_1}{H_1 + H_2} \]

and

\[ K_{m1} = \frac{2H_{m1}}{\omega_0} \omega_2^2 = -K_{12} \frac{(H_1 + H_2)^2}{H_1H_2} \]

The modal equivalent of the generator rotor is

\[ \frac{2H_{m2}}{\omega_0} \frac{d^2\theta_{m2}}{dt^2} + K_{m2}\theta_{m2} = \Delta T_e \]  

(9.27)

where

\[ H_{m2} = \frac{H_2}{H_2} \frac{H_1}{H_1 + H_2} = \frac{H_2}{H_1} \frac{H_1}{H_1 + H_2} \]

and

\[ K_{m2} = \frac{2H_{m2}}{\omega_0} \omega_2^2 = K_{12} \frac{(H_1 + H_2)^2}{H_1^2} \]

The responses of the turbine and the generator rotor due to an excitation \( \Delta T_e \), at a frequency close to \( \omega_2 \), are therefore

\[ \theta_1 = -\frac{\omega_0}{2(H_1 + H_2)} \frac{\Delta T_e}{\omega_2^2 - \omega_2^2} \]  

(9.28)

and

\[ \theta_2 = \frac{H_1}{H_2} \frac{\omega_0}{2(H_1 + H_2)} \frac{\Delta T_e}{\omega_2^2 - \omega_2^2} \]  

(9.29)

Note that the modal parameters for the turbine and hence its response are not affected by interchanging the values \( H_1 \) and \( H_2 \), whereas those for the generator rotor are affected greatly depending on the relative magnitudes of \( H_1 \) and \( H_2 \). For example, comparing the case where \( H_1 = 1, H_2 = 2 \) with that where \( H_1 = 2, H_2 = 1 \), it is seen that the amplitude of oscillation in the second case will be larger by a factor of 4. Therefore, the severity of torsional oscillations can be expected to be less when the generator rotor inertia is relatively high, as in the case of hydro-generators.

The mode shapes for the two cases are shown in Figure 9.3
From equations (9.28) and (9.29), the relative displacement of the generator rotor with respect to the turbine is

$$\theta_{21} = \theta_2 - \theta_1 = \frac{\omega_0}{2H_2} \frac{\Delta T_e}{\omega_1^2 - \omega_2^2}$$

and the shaft torque is

$$K_{21} \theta_{21} = \frac{\omega_0}{2H_2} K_{12} \frac{\Delta T_e}{\omega_1^2 - \omega_2^2}$$

**Damping**

The torsional modes and mode shapes determined on the basis of zero damping will be close to their actual values, since the overall mechanical damping is small. It is impractical to determine the damping coefficients of equation (9.17) from either design data or tests. A good estimate of damping of the individual modal equivalent (equation 9.23) can, however, be obtained from test. This can be done by exciting the torsional system corresponding to a particular mode (say the $k$th mode) and observing the rate of decay of the resulting oscillation.

In the presence of damping, equation (9.23) will be modified to

$$\frac{2H_{mi}}{\omega_o} \frac{d^2 x_{mi}}{dt^2} + \frac{D_{mi}}{\omega_o} \frac{dx_{mi}}{dt} + K_{mi} x_{mi} = \Delta T_e$$

(9.30)

The free response is of the form

$$x_{mi} = A e^{-\alpha t} \cos(\beta t + \theta)$$
TURBINE-GENERATOR SHAFT TORSIONALS

where

\[ \alpha = \frac{D_{mi}}{4H_{mi}}, \quad \beta = \sqrt{\frac{K_{mi} \omega_o}{2H_{mi}}} \]

The free response is illustrated in Figure 9.4

![Figure 9.4 Typical free response of system represented by equation 9.30.](image)

It is customary to express damping in terms of logarithmic decrement which is defined as the natural logarithm of the ratio of two successive peaks, or, from Figure 9.4,

\[ \text{Log Dec} = \ln \frac{\varepsilon^{-\alpha_1}}{\varepsilon^{-\alpha_2}} = \frac{1}{n-1} \ln \frac{\varepsilon^{-\alpha_1}}{\varepsilon^{-\alpha_2}} = \alpha \frac{2\pi}{\omega_k} \]

\[ \therefore D_{mi} = \frac{2\omega_k H_{mi}}{\pi} \text{ Log Dec} \quad (9.31) \]

**Evaluation of Torsional Stress and Fatigue in Turbine-Generator Shafts**

Turbine-generators are designed to withstand, without damage, a limited number of terminal three phase short circuits during their lifetime. The duty associated with major out-of-phase synchronizations is also recognized and turbine-generators are designed to withstand these without gross failures, although significant fatigue life may be lost following each incident. Both of these are, however, relatively rare occurrences. The more commonly encountered disturbances producing abnormal stresses in the turbine-generator shafts are: single and multiple phase faults followed by three phase clearing, fast reclosing, line switching, faulty synchronizing and load rejection. The mechanism of high shaft torque production following any of these incidents has been explained earlier. The actual magnitudes of the peak torques depend on many factors, some of which are not predictable.

The situation is somewhat different when series capacitors are present in transmission lines. Although high shaft torques can be produced following system disturbances in the same way as in systems without series capacitors, the presence of series capacitors constitutes a special hazard, because of the possibility of torque magnification due to interaction between the torsional system and the electrical system. This will be discussed in more detail later.

An accurate calculation of shaft torques is possible by simultaneously solving the equations describing the shaft system, equations (9.11) - (9.15), along with the equations describing the electrical system comprising the generator and the system network. In these calculations a more detailed representation of the generator and system network than is usually required for conventional stability calculation is necessary, in order to capture all the higher frequency
electrical transients that affect the shaft torsional oscillations but have little effect on conventional stability results. The detailed generator and network modeling is described later. Although a simple network configuration has been used in the illustration, the modeling can be extended to more complex networks.

In the absence of series capacitors and excitation control incorporating power system stabilizer employing speed feedback, there is little interaction between the shaft mechanical system and the electrical system. Therefore the torsional and electrical system equations can be solved separately. The generator and network equations can be solved corresponding to the disturbances being investigated, and the resulting air-gap torque can then be used as input to the shaft system equations. The shaft torques obtained in this way will differ very little from those obtained from a simultaneous solution of the mechanical and electrical system. The advantage is a reduction in computational complexity. When the network disturbance and the period over which the computation is to be extended are such that considerable rotor movement occurs, the conventional swing equation (9.16) should be included in the electrical system equations.

In situations where significant interaction between mechanical and electrical system exists, the above computation based on one-way coupling is not valid, and the mechanical and electrical system equations must be solved simultaneously.

**Estimation of fatigue**

Once the response torque has been calculated for the various shaft spans, fatigue life expenditure can be estimated provided that reliable information on fatigue strength characteristics of the shafts is available. Unfortunately, due to the many complexities and uncertainties involved in determining shaft fatigue capability under cyclic torsional stresses, an assessment of the loss of life with a high degree of confidence is difficult, although significant progress has been made in recent years due to concerns for high speed reclosures and subsynchronous resonance.

A valid fatigue model should take into consideration the effects of stress concentration, non-linear stress-strain characteristics, material behavior to cyclic deformation, geometry and size, occasional overstrain, processing, etc.

A typical torsional fatigue characteristic is shown in Figure 9.5. Instead of a single curve a band with an upper and a lower bound curve has been used in order to illustrate the variation in fatigue strength that is possible due to the effect of surface finish, corrosion, surface damage due to
handling and manufacturing process, lack of homogeneity in the shaft material, porosity, impurities, forging tears etc. It is evident that the predicted value of the number of cycles to failure can vary considerably depending on whether the upper or the lower curve is used. The uncertainty is more pronounced in the high cycle fatigue domain. There is considerable difference in opinion in the definition of shaft failure. Failure can be defined as an outright failure, or as reaching the point where cracks appear, necessitating inspection, resurfacing or replacement. The definition of failure accepted by the industry seems to be crack initiation. 100% fatigue implies that crack initiation has been reached.

Any transient event that produces shaft torques exceeding the high cycle fatigue strength will result in some loss of life. The cumulative effect of the number of cycles exceeding the fatigue strength will determine the fatigue damage for the whole event. The fatigue damage for each cycle can be determined using a fatigue life expenditure curve such as the one shown in Figure 9.4. Corresponding to the peak value of the torque during the torque cycle, the number of cycles to failure is read from the curve. The inverse of this number multiplied by one hundred is the percent loss of fatigue life for that torque cycle. The cumulative life expenditure for the completed incident is obtained from the sum total of the life expenditures for the individual cycles. The procedure is to be repeated for each shaft section.

A representative assortment of disturbances based on historic data could be considered for the assessment of the shaft torsional stresses and the accompanying loss of fatigue life during the operating life of a turbine-generator. Due to the many uncertainties involved and the lack of a thorough understanding of the subject of material fatigue, the life expenditure curves supplied by the manufacturers can be expected to be conservative. Therefore, if the estimated loss of life is well below 100%, then, barring extraordinary events, shaft failure can be considered unlikely. In a marginal situation or where shaft failure is indicated, the case should be studied in more detail with the manufacturer.

**Effect of damping**

The amount of damping present in the torsional oscillations determine the number of fatigue cycles experienced by each turbine-generator shaft section following a transient event. It has been recognized that the decay rates of torsional oscillations at natural frequencies are very small. Due to the complexities of the damping mechanisms, it is difficult to reliably predict the damping values from design data. Different damping values have been observed in tests conducted on nominally identical units under similar operating conditions. Tests conducted on a number of turbine-generators have shown the decay time constants of the various oscillation modes to be in the range from 5 to 15 seconds. Tests have also shown the damping values to be functions of generator power output and network configuration -- in general, damping tends to go down as the power output is reduced.

Since the oscillation frequencies are high and the damping is low, the initial peaks of the torsional oscillations are not affected by damping. However, the amount of damping present determines the number of cycles the shafts experience before the amplitudes fall below the high cycle fatigue levels. Estimates of shaft fatigue life expenditure will therefore be critically dependent on the damping assumptions. Since the damping values cannot be precisely determined at the design stage, manufacturers recommend that station tests be performed to accurately measure the damping values, if torsional response evaluations show significant fatigue duty for disturbances that occur relatively frequently on the system.
Series Compensation and Subsynchronous Instability

The presence of series capacitors in transmission systems creates oscillatory electrical circuits that have natural frequencies in the subsynchronous (i.e., below fundamental frequency) range. Consider a synchronous generator connected through a series compensated transmission line to a large power system. If $X_C$ is the per unit capacitive reactance, and $X_L$ is the per unit inductive reactance, the electrical natural frequency is given by

$$f_e = f_o \sqrt{\frac{X_C}{X_L}}$$  (9.32)

where $f_o$ is the rated synchronous frequency (60 Hz). For practical levels of compensation $\frac{X_C}{X_L} < 1$, and therefore $f_e < f_o$.

The subsynchronous oscillations arising from series compensation can, in certain situations, give rise to phenomena leading to instability and/or equipment damage. Before detailed mathematical analyses are undertaken, an introductory discussion of these phenomena will be in order.

Transient torque amplification

In a series compensated system, a large system disturbance, such as a system fault that does not spark over the capacitor protective gap, will cause transient current at electrical natural frequencies (subynchronous) to flow. The frequency of the currents during the fault will depend on the location of the fault and could have any value within a fairly wide range. It is easy to verify that the transient subsynchronous currents in the three phases are unbalanced and therefore will have both positive and negative sequence components. The positive sequence current flowing in the generator armature will interact with the air-gap flux rotating at synchronous speed and produce an electrical air-gap torque at the complementary frequency ($f_o - f_e$). Assuming that the per unit air-gap flux is close to unity, the transient torque will be proportional to the per unit subsynchronous current. Therefore, the magnitude of the torque can be considerable. If the frequency of the torque is close to one of the shaft natural frequencies, great amplification of this torque may result. As illustrated in equation (9.10), shaft torques under resonant conditions can reach high values very quickly.

In the absence of resonance, or if capacitors are bypassed during the fault, the shaft system will still undergo high amplitude oscillations due to the large electrical torques generated by the fault. During the fault the series capacitors in the unfaulted lines will be charged to high voltages. If these voltages are below the capacitor protective gap sparkover level, a large amount of energy will be stored in the $L$-$C$ circuit. After the fault is cleared, the excess energy will discharge, driving considerable transient subsynchronous currents through the generator. The frequencies of these subsynchronous currents are usually predictable since the post-fault network configuration is known. Torque amplification will result if the frequency of the resulting air-gap torque matches one of the torsional natural frequencies. Note that since the shaft system is already in a state of oscillation initiated by the fault, torque amplification after fault clearing will be greater than that during the fault.

The transient subsynchronous torque will decay more slowly than would be calculated from normal circuit inductance and resistance due to the contribution of a negative resistance by the
generator at subsynchronous frequencies. In some situations, as will be seen later, it may even grow.

The negative sequence subsynchronous currents flowing through the generator armature will produce air-gap torque at a supersynchronous frequency \((f_0 + f_e)\). Torsional natural frequencies calculated from the lumped mass model representation of the turbine-generator shaft system as shown in Figure 9.2 are all subsynchronous. The simplified model is adequate if the objective is to assess the possibility of shaft damage, since the model accurately determines the torsional modes in which maximum shaft stresses occur. The simplified model neglects the torsional modes involving deformations of the rotor bodies as well as the additional modes due to the many flexible members, e.g., turbine bladeings attached to the rotor bodies. Calculations based on a more detailed model in which the rotors and shafts are represented by a large number of elements, as well as field tests on completely assembled units, show the presence of numerous natural frequencies extending to 150 Hz. Supersynchronous electrical torques may therefore produce high responses in some turbine-generator components. Turbine-generators are designed to keep natural frequencies well away from 60 Hz and 120 Hz, so as to avoid resonance and consequent torque amplification due to the 60 Hz and 120 Hz electrical torques that result from balanced and unbalanced faults normally occurring in systems without series compensation. The presence of series compensation may require consideration of all the natural frequencies in the supersynchronous range.

**Self excitation**

A power system is never in an absolutely steady state. In a series compensated system any perturbation will produce transient subsynchronous currents of both positive and negative sequence. The positive sequence subsynchronous current flowing in the generator armature will set up a magnetic field which will rotate in the same direction as the generator rotor but at a speed corresponding to the subsynchronous frequency. The generator will therefore act like an induction generator, feeding energy into the system.

From induction machine theory, the generator impedance to the positive sequence subsynchronous current will have a negative resistance component \(R_r/s\), where \(R_r\) is the rotor resistance referred to the armature, and \(s\) is the per unit negative slip given by

\[
s = \frac{\omega_e - \omega_o}{\omega_o}
\]  

(9.33)

If the magnitude of the negative resistance is greater than the net positive resistance of the generator armature and the transmission system, the circuit will become self excited and the subsynchronous current will grow.

Since there is no negative resistance associated with the negative sequence current, it will always damp out.

**Torsional interaction**

Any perturbation, whether mechanical or electrical, will excite torsional natural frequency oscillations of the generator shaft system. An oscillatory rotor gives rise to voltages and currents in the armature circuits at frequencies which are complements of the rotor oscillating frequency. This can be seen from the following analysis.
Consider a generator whose rotor has a small amplitude oscillation superimposed on the steady synchronous speed.

\[
\Delta \omega = A \sin \omega_n t \\
\Delta \theta = -\frac{A}{\omega_n} \cos \omega_n t
\]

From the synchronous machine equations given in Chapter 5 and repeated later in this chapter, the changes in the \(d-q\) components of the terminal voltage due to speed change only (i.e., retaining only the speed voltage terms in the voltage equations) are given by

\[
\Delta e_d = -\frac{\psi_d}{\omega_o} \Delta \omega, \quad \Delta e_q = \frac{\psi_d}{\omega_o} \Delta \omega
\]

The phase a voltage is

\[
e_a = e_d \cos \theta - e_q \sin \theta
\]

\[
\therefore \Delta e_a = \Delta e_d \cos \theta - \Delta e_q \sin \theta \Delta \theta - \Delta e_q \sin \theta - e_q \cos \theta \Delta \theta
\]

Noting that \(\theta = \omega_o t\), where \(\omega_o\) is the rated angular frequency of the system, we have

\[
\Delta e_a = \left[ -\frac{\psi_q}{\omega_o} A \sin \omega_n t \cos \omega_o t + e_d \frac{A}{\omega_n} \cos \omega_n t \sin \omega_o t \\
-\frac{\psi_d}{\omega_o} A \sin \omega_n t \sin \omega_o t - e_q \frac{A}{\omega_n} \cos \omega_n t \cos \omega_o t \right]
\]

In the steady state

\[
e_d \approx -\psi_q, \quad e_q \approx \psi_d
\]

\[
\therefore \Delta e_a = \frac{\omega_o + \omega_n}{\omega_o \omega_n} A V_i \cos[(\omega_o + \omega_n)t + \phi] + \frac{\omega_o - \omega_n}{\omega_o \omega_n} A V_i \cos[(\omega_o - \omega_n)t + \phi]
\]

Equation (9.34) shows that the armature voltage induced due to the rotor oscillating at frequency \(\omega_n\) will have components at frequencies \((\omega_o + \omega_n)\) (supersynchronous), and \((\omega_o - \omega_n)\) (subsynchronous). These voltages will drive currents through the generator armature and the transmission system at the corresponding frequencies. The subsynchronous frequency current can be considerable even for a small induced voltage if the frequency is close to the (subsynchronous) natural frequency of the electrical system. The armature currents will produce pulsating air-gap torques of frequency \(\omega_n\). Under certain condition, as shown later by detailed mathematical analysis, the torque produced by the subsynchronous current can reinforce the rotor oscillations so that the oscillations will grow.

Note that torsional interaction and shaft torque magnification are also possible in the presence of certain excitation and governor control systems, with or without series capacitors in the system. For example, in the application of power system stabilizer employing speed feedback, depending
on the location of the speed sensor on the shaft system, the signal may contain an appreciable amount of one or more shaft system natural frequency components. These signals acting through the excitation system can produce air-gap torques at the shaft natural frequencies, thereby reinforcing the shaft system oscillations. This can be minimized or eliminated, either by positioning the speed sensor close to a suitable node point (the point at which the shaft motion contains zero or minimum amount of a particular frequency component as indicated by the particular mode shape), or by generating the signal indirectly from quantities, such as, accelerating power, that are relatively free of these frequencies.

**Development of Detailed Electrical System Model**

An exact analysis in which the impact of electrical transients, self-excitation and torsional interaction are included simultaneously, can be undertaken by combining the shaft system model with that for the electrical system. However, a more detailed model for the electrical system than is required for conventional stability studies would be needed. A lumped parameter model can still be used. However, the terms that give rise to electrical transients at normal system (power) frequency and electrical natural frequencies must be included. A single-machine model in which a generator is connected to a large system (infinite bus) through a transmission line is usually sufficient, unless the generator being studied is closely coupled electrically with other generators. This is because the high frequency transients are usually short lived, and tend to be localized unlike the low frequency transients involving rotor swings. (The high frequency transients do not generally affect the rotor swing mode appreciably -- hence the justification of neglecting these in stability studies.)

The system to be considered is shown in Figure 9.6.

The system to be considered is shown in Figure 9.6.

![Fig. 9.6 A generator connected to a large system through a series compensated transmission line. The transmission line is represented by an equivalent lumped parameter model. For generality, a step-down transformer has been added at the far end.](image)

A detailed synchronous machine model has been presented in Chapter 5. The relevant equations, assuming one damper winding on each axis, are listed below.

Flux linkage equations:

\[
\psi_d = -x_d i_d + x_{ad} i_{fd} + x_{ad} i_{ld}
\]

\[
\psi_q = -x_q i_q + x_{aq} i_{iq}
\]

\[
\psi_{fd} = -x_{ad} i_d + x_{fa} i_{fd} + x_{f1d} i_{ld}
\]

\[
\psi_{ld} = -x_{ad} i_d + x_{fa} i_{fc} + x_{f1d} i_{ld}
\]

\[
\psi_{iq} = -x_{aq} i_q + x_{11q} i_{ld}
\]

\[(9.35)\]
Voltage equations:

\[ e_d = \frac{1}{\omega_0} \frac{d\psi_d}{dt} - \frac{\omega}{\omega_0} \psi_q - r_i d \]

\[ e_q = \frac{1}{\omega_0} \frac{d\psi_d}{dt} + \frac{\omega}{\omega_0} \psi_d - r_i q \]

\[ e_{fd} = \frac{1}{\omega_0} \frac{d\psi_{fd}}{dt} + r_f i_{fd} \] \hspace{1cm} (9.36)

\[ 0 = \frac{1}{\omega_0} \frac{d\psi_{id}}{dt} + r_i d i_{id} \]

\[ 0 = \frac{1}{\omega_0} \frac{d\psi_{id}}{dt} + r_i q i_{iq} \]

Air-gap torque:

\[ T_e = \psi_d i_q - \psi_q i_d \] \hspace{1cm} (9.37)

Equations for the transmission system:

The various voltage and current quantities and the circuit parameters are shown on Figure 9.6 using subscripts associating them with the various sections. The equations, in phase quantities, for the section between the machine terminal and the high-voltage bus of the step-up transformer can be written as

\[ e_a = R_{i1} i_a + L_{i1} \frac{d i_a}{dt} + E_{1a} \]

\[ e_b = R_{i1} i_b + L_{i1} \frac{d i_b}{dt} + E_{1b} \] \hspace{1cm} (9.38)

\[ e_c = R_{i1} i_c + L_{i1} \frac{d i_c}{dt} + E_{1c} \]

where \( e_a, e_b, e_c \) are the instantaneous phase voltages at the generator terminal etc. The equations can be written in matrix form as

\[ \mathbf{e}_{abc} = R_{i1} \mathbf{i}_{abc} + L_{i1} \frac{d}{dt} \mathbf{i}_{abc} + \mathbf{E}_{1,abc} \] \hspace{1cm} (9.39)

where

\[ \mathbf{e}_{abc} = \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} \text{ etc.} \]

Similarly,

\[ \mathbf{I}_{sh1,abc} = C_{sh} \frac{d}{dt} \mathbf{E}_{1,abc} \] \hspace{1cm} (9.40)

\[ \mathbf{E}_{1,abc} = R \mathbf{I}_{L,abc} + L \frac{d}{dt} \mathbf{I}_{L,abc} + \mathbf{v}_{c,abc} + \mathbf{E}_{2,abc} \] \hspace{1cm} (9.41)

\[ \mathbf{I}_{L,abc} = C \frac{d}{dt} \mathbf{v}_{c,abc} \] \hspace{1cm} (9.42)
TURBINE-GENERATOR SHAFT TORSIONALS

\[ I_{sh2,abc} = C_{sh} \frac{d}{dt} E_{2,abc} \quad (9.43) \]

\[ E_{2,abc} = R_{i2} I_{S,abc} + L_{i2} \frac{d}{dt} I_{S,abc} + V_{b,abc} \quad (9.44) \]

Also

\[ i_{abc} = I_{sh1,abc} + I_{L,abc} \quad (9.45) \]

\[ I_{L,abc} = I_{sh2,abc} + I_{S,abc} \quad (9.46) \]

Using the same transformation as used in transforming the synchronous machine phase quantities into \(d-q-o\) quantities, the system phase quantities can also be transformed into \(d-q-o\) quantities. The transformation is given by, in the case of currents for example (see Chapter 5),

\[ i_{dqo} = T i_{abc} \quad (9.47) \]

where

\[
T = \frac{2}{3} \begin{bmatrix}
\cos \theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\
-\sin \theta & -\sin(\theta - 120^\circ) & -\sin(\theta + 120^\circ) \\
1/2 & 1/2 & 1/2
\end{bmatrix}
\]

The inverse transformation is

\[ i_{abc} = T^{-1} i_{dqo} \quad (9.48) \]

where

\[
T^{-1} = \begin{bmatrix}
\cos \theta & -\sin \theta & 1 \\
\cos(\theta - 120^\circ) & -\sin(\theta - 120^\circ) & 1 \\
\cos(\theta + 120^\circ) & -\sin(\theta + 120^\circ) & 1
\end{bmatrix}
\]

Applying these transformations to the above equations we obtain

\[ e_{dqo} = R_{i1} i_{dqo} + L_{i1} \frac{d}{dt} i_{dqo} + \omega L_{i1} S i_{dqo} + E_{1,dqo} \quad (9.49) \]

since

\[ T \frac{d}{dt} (T^{-1} i_{dqo}) = \frac{d}{dt} i_{dqo} + \frac{d\theta}{dt} S i_{dqo} = \frac{d}{dt} i_{dqo} + \omega S i_{dqo} \]

where

\[
S = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Similarly,

\[ I_{sh1,dqo} = C_{sh} \frac{d}{dt} E_{1,dqo} + \omega C_{sh} S E_{1,dqo} \quad (9.50) \]

\[ E_{1,dqo} = R \ P I_{L,dqo} + L \frac{d}{dt} I_{L,dqo} + \omega L S I_{L,dqo} + v_{C,dqo} + E_{2,dqo} \quad (9.51) \]

\[ I_{L,dqo} = C \frac{d}{dt} v_{C,dqo} + \omega C S v_{C,dqo} \quad (9.52) \]
TURBINE-GENERATOR SHAFT TORSIONALS

\[ I_{sh2, dqo} = C_{sh} \frac{d}{dt} E_{2, dqo} + \omega C_{sh} S E_{2, dqo} \]  
\[ E_{2, dqo} = R_{i2} I_{S, dqo} + L_{i2} \frac{d}{dt} I_{S, dqo} + \omega L_{i2} S I_{S, dqo} + V_{b, dqo} \]  

Also
\[ i_{dqo} = I_{sh1, dqo} + I_{L, dqo} \]  
\[ i_{L, dqo} = I_{sh2, dqo} + I_{S, dqo} \]  

When converted into per-unit form the above equations reduce to
\[ e_{dqo} = R_{i1} i_{dqo} + \frac{1}{\omega_o} X_{i1} \frac{d}{dt} i_{dqo} + \frac{\omega}{\omega_o} X_{i1} S i_{dqo} + E_{1, dqo} \]  
\[ I_{sh1, dqo} = \frac{1}{\omega_o} Y \frac{d}{dt} E_{1, dqo} + \frac{\omega}{\omega_o} S E_{1, dqo} \]  
\[ E_{1, dqo} = R I_{L, dqo} + \frac{1}{\omega_o} X \frac{d}{dt} I_{L, dqo} + \frac{\omega}{\omega_o} X S I_{L, dqo} + v_{c, dqo} + E_{2, dqo} \]  
\[ I_{L, dqo} = \frac{1}{\omega_o} \frac{1}{X_c} \frac{d}{dt} v_{c, dqo} + \frac{\omega}{\omega_o} S v_{c, dqo} \]  
\[ I_{sh2, dqo} = \frac{1}{\omega_o} Y \frac{d}{dt} E_{2, dqo} + \frac{\omega}{\omega_o} S E_{2, dqo} \]  
\[ E_{2, dqo} = R_{i2} I_{S, dqo} + \frac{1}{\omega_o} X_{i2} \frac{d}{dt} I_{S, dqo} + \frac{\omega}{\omega_o} X_{i2} S I_{S, dqo} + V_{b, dqo} \]  

where \( Y \) is the total admittance in pu due to line charging, and \( X_c \) is the capacitive reactance due to the series capacitor.

Equations (9.55) and (9.56) remain unchanged.

The steady state phasor diagram is shown in Figure 9.7.

Fig. 9.7 Steady-state phasor diagram for system of Fig. 9.6.
Equations (9.55) through (9.62), along with equations (9.35) through (9.37), and equations (9.11) through (9.15) constitute a complete description of the combined turbine-generator shaft torsional and electrical system. For a given disturbance scenario the equations can be solved numerically and the relative angular displacements of the rotor elements determined as functions of time. The shaft torques are then obtained by multiplying these by the corresponding spring constants. The effects of excitation and governor control systems can be included by adding the appropriate equations to be above model.

Since the system response would contain a number of high frequency modes, a very small time step would be needed in the numerical computation in order to maintain accuracy as well as preserve the oscillations in the computed response. The procedure will therefore be extremely time consuming if the simulation has to be continued for a fairly long period of real time which may be necessary in order to identify the presence of self-excitation and torsional interaction.

Small Disturbance Stability Study of Turbine-Generator Shaft Torsional System

The possibility of self-excitation and/or torsional interaction can be ascertained more conveniently through a small disturbance stability analysis, in which the system equations are linearized about an operating point and the stability of the resulting linear system is investigated.

The analysis will first be carried out using a state-space approach, as it offers a more complete information on the stability properties of the system. Later a simplified analysis using a frequency response technique will be illustrated.

Since excitation and governor control systems can contribute to torsional interactions, these should be included in the analysis. For the purpose of illustration, a typical simplified excitation control model is included in the analysis. Inclusion of a governor model is relatively straightforward and is therefore not considered here.

The machine flux linkage equations (9.35), after linearization, can be written in matrix form as

\[
\begin{bmatrix} \Delta \Psi_s \\ \Delta \Psi_r \end{bmatrix} = X \begin{bmatrix} \Delta i_s \\ \Delta i_r \end{bmatrix} \quad (9.63)
\]

where

\[
\Delta \Psi_s = \begin{bmatrix} \Delta \psi_d \\ \Delta \psi_q \end{bmatrix}, \quad \Delta \Psi_r = \begin{bmatrix} \Delta \psi_{fd} \\ \Delta \psi_{f1d} \\ \Delta \psi_{q1} \end{bmatrix}, \quad \Delta i_s = \begin{bmatrix} \Delta i_d \\ \Delta i_{1d} \end{bmatrix}, \quad \Delta i_r = \begin{bmatrix} \Delta i_{fd} \\ \Delta i_{f1d} \\ \Delta i_{1q} \end{bmatrix}
\]

and

\[
X = \begin{bmatrix}
-x_d & x_{ad} & x_{ad} & x_{ag} \\
-x_q & -x_d & x_{ad} & x_{ag} \\
-x_{ad} & x_{fd} & x_{f1d} & x_{11d} \\
-x_{ag} & -x_{ad} & x_{f1d} & x_{11d} \\
-x_{ag} & -x_{ad} & -x_{ag} & x_{11q}
\end{bmatrix}
\]

From (9.63)

\[
\begin{bmatrix} \Delta i_s \\ \Delta i_r \end{bmatrix} = Y \Delta \Psi \quad (9.64)
\]
where \( Y \) is the inverse of \( X \) and

\[ \Delta \Psi = \begin{bmatrix} \Delta \Psi_s \\ \Delta \Psi_r \end{bmatrix} \]

From (9.64), after partitioning \( Y \) as

\[ \Delta i_s = P \Delta \Psi \]

\[ \Delta i_r = Q \Delta \Psi \]  \hspace{1cm} (9.65)

\hspace{1cm} (9.66)

The machine voltage equations, after linearization, can be written in matrix form as

\[
\frac{d}{dt} \Delta \Psi_s = \begin{bmatrix} -\omega_o & \omega_o \\
-\psi_d & -\psi_d \end{bmatrix} \Delta \Psi_s + \begin{bmatrix} \psi_q \\
-\psi_d \end{bmatrix} \Delta \omega + \omega_o r \Delta i_s + \omega_o \begin{bmatrix} \Delta e_d \\
\Delta e_q \end{bmatrix} \hspace{1cm} (9.67)
\]

\[
\frac{d}{dt} \Delta \Psi_r = A \Delta i_r + \begin{bmatrix} x_{f/q} \Delta E_{f/q} \\
x_{a/q} T_{do}' \end{bmatrix} \hspace{1cm} (9.68)
\]

where

\[ A = \text{diag} \left[ -\frac{x_{f/q}}{T_{do}} - \omega_o r_{id} - \omega_o r_{iq}, \ 0 \right], \quad E_{f/q} = \frac{x_{a/q}}{r_{f/q}} e_{f/q} \quad \text{and} \quad T_{do}' = \frac{x_{f/q}}{\omega_o r_{f/q}} \]

The air gap torque equation (9.37) can be written in matrix form as

\[
\Delta T_e = \begin{bmatrix} i_q - i_d \end{bmatrix} \Delta \Psi_s + \begin{bmatrix} -\psi_d & \psi_d \end{bmatrix} \Delta i_s \hspace{1cm} (9.69)
\]

Using (9.65) and (9.66), equations (9.67), (9.68) and (9.69) reduce to

\[
\frac{d}{dt} \Delta \Psi_s = \begin{bmatrix} -\omega_o & \omega_o \\
-\psi_d & -\psi_d \end{bmatrix} \Delta \Psi_s + \begin{bmatrix} \psi_q \\
-\psi_d \end{bmatrix} \Delta \omega + \omega_o r \Delta \Psi + \omega_o \begin{bmatrix} \Delta e_d \\
\Delta e_q \end{bmatrix} \hspace{1cm} (9.70)
\]

\[
\frac{d}{dt} \Delta \Psi_r = A Q \Delta \Psi + \begin{bmatrix} x_{f/q} \Delta E_{f/q} \\
x_{a/q} T_{do}' \end{bmatrix} \hspace{1cm} (9.71)
\]

\[
\Delta T_e = \begin{bmatrix} i_q - i_d \end{bmatrix} \Delta \Psi_s + \begin{bmatrix} -\psi_d & \psi_d \end{bmatrix} P \Delta \Psi \hspace{1cm} (9.72)
\]

The transmission system equations can be linearized and rearranged, noting that for balanced operation the zero-sequence quantities are absent, as follows:
TURBINE-GENERATOR SHAFT TORSIONALS

From (9.57)

\[
\begin{bmatrix}
\Delta e_d \\
\Delta e_q
\end{bmatrix} = \frac{X_{rl}}{\omega_o} \frac{d}{dt} \Delta i_s + \begin{bmatrix}
R_{rl} & -X_{rl} \\
X_{rl} & R_{rl}
\end{bmatrix} \Delta i_s + \frac{X_{rl}}{\omega_o} \begin{bmatrix}
i_d & -i_q
\end{bmatrix} \Delta \omega + \begin{bmatrix}
\Delta E_{id} \\
\Delta E_{iq}
\end{bmatrix}
\]

which can be written as, using equation (9.65) and writing \(\Delta E_1 = \begin{bmatrix}\Delta E_{1d} \\ \Delta E_{1q}\end{bmatrix}\),

\[
\begin{bmatrix}
\Delta e_d \\
\Delta e_q
\end{bmatrix} = \frac{X_{rl}}{\omega_o} \frac{d}{dt} \Delta \Psi + \begin{bmatrix}
R_{rl} & -X_{rl} \\
X_{rl} & R_{rl}
\end{bmatrix} \Delta \Psi + \frac{X_{rl}}{\omega_o} \begin{bmatrix}
i_d & -i_q
\end{bmatrix} \Delta \omega + \Delta E_1 \tag{9.73}
\]

From (9.55) and (9.58), and using (9.65)

\[
\frac{d}{dt}\Delta E_1 = \frac{2\omega_o}{Y} \Delta \Psi - \frac{2\omega_o}{Y} \Delta I_L + \begin{bmatrix}
\omega_o & 0 \\
-\omega_o & \omega_o
\end{bmatrix} \Delta E_1 + \begin{bmatrix}
E_{eq} \\
-\omega_o E_{id}
\end{bmatrix} \Delta \omega \tag{9.74}
\]

where

\[
\Delta I_L = \begin{bmatrix}
\Delta I_{ld} \\
\Delta I_{lq}
\end{bmatrix}
\]

Similarly, from (9.56), and (9.59) through (9.62)

\[
\frac{d}{dt}\Delta I_L = \frac{\omega_o}{X} \Delta E_1 + \omega_o \begin{bmatrix}
-R/X & 1 \\
-1 & -R/X
\end{bmatrix} \Delta I_L + \begin{bmatrix}
I_{eq} \\
-I_{id}
\end{bmatrix} \Delta \omega - \frac{\omega_o}{X} \Delta v_C - \frac{\omega_o}{X} \Delta E_2 \tag{9.75}
\]

where

\[\Delta v_C = \begin{bmatrix}
\Delta v_{cd} \\
\Delta v_{cq}
\end{bmatrix}\] and \(\Delta E_2 = \begin{bmatrix}\Delta E_{2d} \\ \Delta E_{2q}\end{bmatrix}\)

\[
\frac{d}{dt}\Delta v_C = \omega_o X_C \Delta I_L + \begin{bmatrix}
\omega_o & 0 \\
-\omega_o & \omega_o
\end{bmatrix} \Delta v_C + \begin{bmatrix}
v_{cq} \\
v_{cd}
\end{bmatrix} \Delta \omega \tag{9.76}
\]

\[
\frac{d}{dt}\Delta E_2 = \frac{2\omega_o}{Y} \Delta I_L - \frac{2\omega_o}{Y} \Delta I_s + \begin{bmatrix}
\omega_o & 0 \\
-\omega_o & \omega_o
\end{bmatrix} \Delta E_2 + \begin{bmatrix}
E_{eq} \\
-E_{2d}
\end{bmatrix} \Delta \omega \tag{9.77}
\]

where

\[
\Delta I_s = \begin{bmatrix}
\Delta I_{sd} \\
\Delta I_{sq}
\end{bmatrix}
\]

\[
\frac{d}{dt}\Delta I_s = \frac{\omega_o}{X_{r2}} \Delta E_2 + \omega_o \begin{bmatrix}
-R_{r2}/X_{r2} & 1 \\
-1 & -R_{r2}/X_{r2}
\end{bmatrix} \Delta I_s + \begin{bmatrix}
I_{sq} \\
-I_{sd}
\end{bmatrix} \Delta \omega + \frac{\omega_o}{X_{r2}} \frac{V_p}{\omega_o} \begin{bmatrix}
-\cos \delta \\
\sin \delta
\end{bmatrix} \Delta \delta \tag{9.78}
\]

Note that \(\Delta \Psi, \Delta E_1, \Delta I_L, \Delta v_C, \Delta E_2, \) and \(\Delta I_s\) are vectors of state variables as defined. Therefore, equations (9.73) through (9.78) are expressed entirely in terms of state variables.
Considering the simplified excitation control system used in Chapter 6 (Fig. 6.6, IEEE type 1), ignoring saturation for the purpose of illustration, the linearized equations can be written in matrix form as (see Appendix C)

\[
\frac{d}{dt} \begin{bmatrix} \Delta E_{fd} \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{K_F K_A}{T_F T_A} + \frac{1}{T_A} & \frac{K_F K_A}{T_F T_A} \\ \frac{1}{T_F} & -\frac{1}{T_F} \end{bmatrix} \begin{bmatrix} \Delta E_{fd} \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{-K_A}{T_A} \\ 0 \end{bmatrix} \Delta V_t \tag{9.79}
\]

Since \( V_t^2 = e_{d}^2 + e_{q}^2 \), \( \Delta V_t \) can be expressed as

\[
\Delta V_t = \begin{bmatrix} e_{d} \\ e_{q} \\ V_t \end{bmatrix} \begin{bmatrix} \Delta e_d \\ \Delta e_q \end{bmatrix}
\]

(9.80)

Therefore, (9.79) can be written as

\[
\frac{d}{dt} \Delta x_{ex} = \mathbf{K}_{ex} \Delta x_{ex} + \mathbf{L} \begin{bmatrix} \Delta e_d \\ \Delta e_q \end{bmatrix}
\]

(9.81)

Substituting for \( \begin{bmatrix} \Delta e_d \\ \Delta e_q \end{bmatrix} \) from (9.73), equations (9.70), (9.71), and (9.81) can be combined into one matrix equation as shown in (9.82)

\[
\frac{d}{dt} \begin{bmatrix} \Delta \Psi \\ \Delta x_{ex} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \Delta \Psi \\ \Delta x_{ex} \end{bmatrix} + \mathbf{C} \frac{d}{dt} \begin{bmatrix} \Delta \Psi \\ \Delta x_{ex} \end{bmatrix} + \mathbf{D} \Delta \omega + \mathbf{G} \Delta E_i
\]

(9.82)

where the matrices \( \mathbf{B}, \mathbf{C}, \mathbf{D} \) and \( \mathbf{G} \) have been obtained by properly combining the constituent matrices of equations (9.70), (9.71), (9.73), and (9.81), and augmenting with zeros as necessary.

Equation (9.82) can be reduced to

\[
\frac{d}{dt} \begin{bmatrix} \Delta \Psi \\ \Delta x_{ex} \end{bmatrix} = [\mathbf{I} - \mathbf{C}]^{-1} \mathbf{B} \begin{bmatrix} \Delta \Psi \\ \Delta x_{ex} \end{bmatrix} + \mathbf{D} \Delta \omega + [\mathbf{I} - \mathbf{C}]^{-1} \mathbf{G} \Delta E_i
\]

(9.83)

The linearized equations of the shaft torsional system, assuming constant mechanical input, can be written in terms of state variables as

\[
\frac{d}{dt} \Delta \theta = \mathbf{K} \Delta \theta + \mathbf{J} \Delta T_c
\]

(9.84)

where

\[
\Delta \theta = \begin{bmatrix} \Delta \theta_1 & \Delta \theta_2 & \cdots & \Delta \theta_5 & \Delta \dot{\theta}_1 & \Delta \dot{\theta}_2 & \cdots & \Delta \dot{\theta}_5 \end{bmatrix}
\]

\[
\mathbf{J} = \begin{bmatrix} 0 & 0 & \cdots & -\frac{\omega_0}{2H_4} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\]
where

\( \mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_3 & \mathbf{K}_4 \end{bmatrix} \)

\( \mathbf{K}_1 \) is a 5\( \times \)5 null matrix

\( \mathbf{K}_2 \) is a 5\( \times \)5 unit matrix

\[
\begin{bmatrix}
-\frac{\omega_0 K_{12}}{2H_1} & -\frac{\omega K_{12}}{2H_1} & \frac{\omega_0 K_{23}}{2H_2} \\
-\frac{\omega_0 K_{12}}{2H_2} & -\frac{\omega_0 (K_{12} + K_{23})}{2H_2} & \frac{\omega_0 K_{23}}{2H_2} \\
\frac{\omega_0 K_{23}}{2H_3} & \frac{\omega_0 (K_{23} + K_{34})}{2H_3} & -\frac{\omega_0 K_{34}}{2H_3} \\
\frac{\omega_0 K_{23}}{2H_4} & \frac{\omega_0 (K_{23} + K_{34})}{2H_4} & -\frac{\omega_0 K_{45}}{2H_4} \\
-\frac{\omega K_{12}}{2H_1} & -\frac{\omega (D_{12} + D_{13})}{2H_1} & \frac{\omega D_{23}}{2H_3} \\
-\frac{\omega_0 (D_{12} + D_{13} + D_{23})}{2H_2} & -\frac{\omega_0 (D_{23} + D_{34})}{2H_3} & \frac{\omega_0 D_{34}}{2H_4} \\
\frac{\omega_0 (D_{23} + D_{34})}{2H_3} & \frac{\omega_0 (D_{34} + D_{45})}{2H_4} & -\frac{\omega_0 (D_{45} + D_{55})}{2H_5}
\end{bmatrix}
\]

Using equation (9.72), (9.84) reduces to

\[
\frac{d\Delta\theta}{dt} = \mathbf{K}\Delta\theta + \mathbf{M}\Delta\Psi
\] (9.85)

Equations (9.83), (9.85) and (9.74) through (9.78) constitute the complete set of state equations describing the system of Figure 9.6. Since these equations are expressed entirely in terms of state variables, they can be readily combined into the form \( \mathbf{x} = \mathbf{A}\mathbf{x} \). Small disturbance stability studies can then be performed as detailed in Chapter 8. The necessary information on small disturbance system performance and stability are obtained from the eigenvalues and eigenvectors of the matrix \( \mathbf{A} \). For stability, the eigenvalues of \( \mathbf{A} \) must have negative real parts.

**Estimation of Electrical Damping in Torsional Interaction**

An analysis of torsional interaction using a frequency response technique will now be presented. We will analyze the system response due to small amplitude rotor oscillations at oscillation frequency \( \omega_n \) corresponding to one of the torsional natural frequencies.

Assuming a small deviation \( \Delta\omega \) of rotor speed from synchronous speed \( \omega_o \),

\[
\Delta\omega = A\sin\omega_o t
\] (9.86)

For sinusoidal oscillations at frequency \( \omega_o \),
\[ \Delta \omega = \frac{d}{dt} \Delta \delta = s \Delta \delta \equiv j \omega_n \Delta \delta \]

or

\[ \Delta \delta = -\frac{j}{\omega_n} \Delta \omega = -j \frac{A}{\omega_n} \sin \omega_n t = -\frac{A}{\omega_n} \cos \omega_n t \]  
(9.87)

The small changes in the generator terminal voltages in terms of \(d-q\) components, \(\Delta e_d\) and \(\Delta e_q\), will, in general, be linear functions of \(\Delta \delta\) and \(\Delta \omega\). The changes in phase voltages can be obtained from equation (9.48). For example, for phase \(a\)

\[ e_a = e_d \cos \theta - e_q \sin \theta \]

Linearizing the above expression, noting that \(\Delta \theta = \Delta \delta\),

\[ \Delta e_a = (\Delta e_d - e_q \Delta \delta) \cos \theta - (\Delta e_q + e_d \Delta \delta) \sin \theta \]  
(9.88)

Also, we can write

\[ \Delta e_d - e_q \Delta \delta = (e_1 + j e_2) \Delta \delta = (e_1 + j e_2) \left(-\frac{A}{\omega_n}\right) \cos \omega_n t \]

and

\[ \Delta e_q + e_d \Delta \delta = (e_3 + j e_4) \Delta \delta = (e_3 + j e_4) \left(-\frac{A}{\omega_n}\right) \cos \omega_n t \]

Therefore, equation (9.88) can be written as, noting that \(\theta = \omega_n t\),

\[ \Delta e_a = -\frac{A}{\omega_n} (e_1 + j e_2) \cos \omega_n t \cos \omega_n t + \frac{A}{\omega_n} (e_3 + j e_4) \cos \omega_n t \sin \omega_n t \]

which reduces to, using (9.87),

\[ \Delta e_a = \Delta E_+ \cos[(\omega_o + \omega_n) t - \phi_+] + \Delta E_- \cos[(\omega_o - \omega_n) t - \phi_-] \]  
(9.89)

where

\[ \Delta E_+ = \frac{A}{2 \omega_o} \sqrt{(-e_1 + e_4)^2 + (e_2 + e_3)^2} \]

\[ \Delta E_- = \frac{A}{2 \omega_o} \sqrt{(-e_1 - e_4)^2 + (-e_2 + e_3)^2} \]

\[ \phi_+ = \tan^{-1} \frac{e_2 + e_3}{-e_1 + e_4}, \quad \phi_- = \tan^{-1} \frac{-e_2 + e_3}{-e_1 - e_4} \]

Note that considering the effects of speed change only, i.e., retaining only the speed voltage terms in the generator voltage equations,

\[ \Delta e_d = \frac{e_d}{\omega_o} \Delta \omega, \quad \Delta e_q = \frac{e_q}{\omega_o} \Delta \omega \]

\[ \therefore \quad e_1 = -e_q, \quad e_2 = \frac{\omega_n}{\omega_o} e_d, \quad e_3 = e_d, \quad e_4 = \frac{\omega_n}{\omega_o} e_q \]
TURBINE-GENERATOR SHAFT TORSIONALS

\[
\Delta E_+ = \frac{A}{2 \omega_o \omega_n} \sqrt{e_d^2 + e_q^2}, \quad \Delta E_- = \frac{A}{2 \omega_o \omega_n} \sqrt{e_d^2 + e_q^2}
\]

\[
\phi_+ = \phi_- = \tan^{-1} \frac{e_d}{e_q}
\]

as derived earlier (see equation (9.34)).

Expressions for phases \(b\) and \(c\) will be similar but with phases displaced by \(-120^\circ\) and \(+120^\circ\), respectively.

Equation (9.89) shows that the phase voltage is composed of two components -- one at supersynchronous frequency \(\omega_o + \omega_n\), and the other at subsynchronous frequency \(\omega_o - \omega_n\). In the above and subsequent expressions the subscripts + and – denote supersynchronous and subsynchronous quantities, respectively.

The supersynchronous and subsynchronous voltages will drive currents at the corresponding frequencies. For phase \(a\) the currents are

\[
\Delta i_{a+} = \frac{\Delta E_+}{Z_{N+}} \cos[(\omega_o + \omega_n)t - \phi_+ - \theta_+]
\]

\[
\Delta i_{a-} = \frac{\Delta E_-}{Z_{N-}} \cos[(\omega_o - \omega_n)t - \phi_- - \theta_-]
\]

and

\[
\Delta i_a = \Delta i_{a+} + \Delta i_{a-}
\]

where \(Z_{N+}\) and \(Z_{N-}\) are the impedances at supersynchronous and subsynchronous frequencies, looking into the network from the generator terminal (the driving point impedances). \(Z_{N+}\) and \(Z_{N-}\) can be obtained from a frequency scan of the system network.

\(\theta_+\) and \(\theta_-\) are the impedance angles.

Similar expressions can be obtained for phases \(b\) and \(c\).

We now transform back to \(d-q\) components. From equation (9.47) we have

\[
e_d = \frac{2}{3} \left[ e_a \cos \theta + e_b \cos(\theta - 120^\circ) + e_c \cos(\theta + 120^\circ) \right]
\]

Linearizing the above expression

\[
\Delta e_d = -\frac{2}{3} \left[ e'_a \sin \theta + e'_b \sin(\theta - 120^\circ) + e'_c \sin(\theta + 120^\circ) \right] \Delta \theta
\]

\[
+ \frac{2}{3} \left[ \Delta e'_a \cos \theta + \Delta e'_b \cos(\theta - 120^\circ) + \Delta e'_c \cos(\theta + 120^\circ) \right]
\]

Using (9.47) and (9.89) the above reduces to

\[
\Delta e_d - e_q \Delta \delta = \Delta E_+ \cos(\omega_n t - \phi_-) + \Delta E_- \cos(\omega_n t + \phi_-)
\]
Similarly,

\[ \Delta e_q + e_d \Delta \delta = \Delta E_+ \sin(\omega_n t - \phi_+) - \Delta E_- \sin(\omega_n t + \phi_-) = -j \Delta E_+ \cos(\omega_n t - \phi_+) + j \Delta E_- \cos(\omega_n t + \phi_-) \]  

(9.94)

The corresponding expressions for currents are

\[ \Delta i_q - i_d \Delta \delta = \frac{\Delta E_+}{Z_{N+}} \cos(\omega_n t - \phi_+ - \theta_+) + \frac{\Delta E_-}{Z_{N-}} \cos(\omega_n t + \phi_- + \theta_-) \]  

(9.95)

and

\[ \Delta i_q + i_d \Delta \delta = -j \frac{\Delta E_+}{Z_{N+}} \cos(\omega_n t - \phi_+ - \theta_+) + j \frac{\Delta E_-}{Z_{N-}} \cos(\omega_n t + \phi_- + \theta_-) \]  

(9.96)

In equations (9.93) - (9.96) the voltages and currents are expressed in terms of supersynchronous and subsynchronous components. Denoting \( \Delta E_+ \cos(\omega_n t - \phi_+) \) by \( E_+ \) and \( \Delta E_- \cos(\omega_n t + \phi_-) \) by \( E_- \), and similarly for currents, it is easy to see that

\[ I_+ = \frac{E_+}{Z_{N+}} \]  

(9.97)

and

\[ I_- = \frac{E_-}{Z_{N-}} \]  

(9.98)

where \( Z_{N-}^* \) is the complex conjugate of \( Z_{N-} \).

Equations (9.97) and (9.98) can be written in matrix form as

\[
\begin{bmatrix}
E_+ \\
E_-
\end{bmatrix} = \begin{bmatrix}
Z_{N+} & Z_{N-}^* \\
Z_{N-} & Z_{N+}^*
\end{bmatrix} \begin{bmatrix}
I_+ \\
I_-
\end{bmatrix}
\]  

(9.99)

Note that \( \Delta e_d - e_d \Delta \delta = E_+ + E_- \) and \( \Delta e_q + e_d \Delta \delta = -jE_+ + jE_- \), from (9.93) and (9.94), and similarly for currents.

Therefore, equation (9.99) is manipulated as follows:

\[
\begin{bmatrix}
1 & 1 \\
-j & j
\end{bmatrix} \begin{bmatrix}
E_+ \\
E_-
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-j & j
\end{bmatrix} \begin{bmatrix}
Z_{N+} & Z_{N-}^* \\
Z_{N-} & Z_{N+}^*
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-j & j
\end{bmatrix} \begin{bmatrix}
I_+ \\
I_-
\end{bmatrix}
\]  

which yields

\[
\begin{bmatrix}
\Delta e_d - e_d \Delta \delta \\
\Delta e_q + e_d \Delta \delta
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2}(Z_{N-}^* + Z_{N+}) & -j \frac{1}{2}(Z_{N-}^* - Z_{N+}) \\
\frac{j}{2}(Z_{N-}^* - Z_{N+}) & \frac{1}{2}(Z_{N-}^* + Z_{N+})
\end{bmatrix} \begin{bmatrix}
\Delta i_d - i_d \Delta \delta \\
\Delta i_q + i_d \Delta \delta
\end{bmatrix}
\]  

(9.100)

Note that in the case of the simple system of one generator connected to an infinite bus through a series compensated transmission line as shown in Figure 9.8,
TURBINE-GENERATOR SHAFT TORSIONALS

Fig. 9.8 A generator connected to an infinite bus through a series compensated transmission line.

\[ Z_{N+} = R + j \frac{\omega_o + \omega_n}{\omega_o} X - j \frac{\omega_o}{\omega_o + \omega_n} X_C \]

\[ Z_{N-}^* = R - j \frac{\omega_o - \omega_n}{\omega_o} X + j \frac{\omega_o}{\omega_o - \omega_n} X_C \]

\[ \frac{1}{2} (Z_{N-}^* + Z_{N+}) = R + j \frac{\omega_n}{\omega_o} \left( X + \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \right) \]

\[ \frac{j}{2} (Z_{N-}^* - Z_{N+}) = X - \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \]

and

\[
\begin{bmatrix}
\Delta e_d - e_q \Delta \delta \\
\Delta e_q + e_q \Delta \delta
\end{bmatrix} = \begin{bmatrix}
R + j \frac{\omega_n}{\omega_o} \left( X + \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \right) & - \left( X - \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \right) \\
\left( X - \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \right) & R + j \frac{\omega_n}{\omega_o} \left( X + \frac{\omega_o^2 X_C}{\omega_o^2 - \omega_n^2} \right)
\end{bmatrix} \begin{bmatrix}
\Delta i_d - i_q \Delta \delta \\
\Delta i_q + i_q \Delta \delta
\end{bmatrix}
\] (9.101)

Equation (9.101) can be derived directly for the simple one-machine system by writing the system equations in \(d-q\) components (given earlier) using the Laplace operator \(s\) for \(d/dt\), linearizing the equations, and then substituting \(j \omega_n\) for \(s\). This is left as an exercise.

We will now develop the necessary equations for the generator. By eliminating the rotor currents the generator stator flux linkages may be expressed as

\[ \psi_d = -x_d \left( f \right) i_d + G \left( f \right) E_{fd} \] (9.102)

\[ \psi_q = -x_q \left( f \right) i_q \] (9.103)

where

\[ x_d \left( f \right) = \frac{1 + T_d' f}{1 + T_d'' f} \left( 1 + T_{qd} f \right) \]

\[ x_q \left( f \right) = \frac{1 + T_q' f}{1 + T_q'' f} \left( 1 + T_{dq} f \right) \]

\[ G \left( f \right) = \frac{1 + T_{kd} f}{1 + T_{kd} f} \left( 1 + T_{qd} f \right) \]

The symbols used in the above expressions have the usual meanings. \(x_d(f)\) and \(x_q(f)\) are known as the direct and quadrature axis operational impedances, respectively.

We will also need the stator voltage and air-gap torque equations, which are re-stated below.
TURBINE-GENERATOR SHAFT TORSIONALS

\[ e_d = \frac{1}{\omega_o} s \psi_d - \frac{\omega}{\omega_o} \psi_q - ri_d \]  \hspace{1cm} (9.104)

\[ e_q = \frac{1}{\omega_o} s \psi_q + \frac{\omega}{\omega_o} \psi_d - ri_q \]  \hspace{1cm} (9.105)

\[ T_e = \psi_d i_q - \psi_q i_d \]  \hspace{1cm} (9.106)

Equations (9.102) - (9.105) can be linearized, assuming fixed field voltage (note that conventional excitation system will not respond to torsional interaction), and written in matrix form as

\[
\begin{bmatrix}
\Delta e_d \\
\Delta e_q
\end{bmatrix} =
\begin{bmatrix}
r + \frac{s}{\omega_o} x_d(s) & -x_q(s) \\
x_d(s) & r + \frac{s}{\omega_o} x_q(s)
\end{bmatrix}
\begin{bmatrix}
\Delta i_d \\
\Delta i_q
\end{bmatrix}
+ \begin{bmatrix}
-\psi_q \\
\psi_d \\
\end{bmatrix}\frac{\omega}{\omega_o}
\]  \hspace{1cm} (9.107)

In the steady state we have, from equations (9.102) to (9.105),

\[ \psi_d = E_{fd} - x_d i_d \hspace{1cm} \psi_q = -x_q i_q \]

\[ e_d = -\psi_q - ri_d \hspace{1cm} e_q = \psi_d - ri_q \]

Therefore, (9.107) can be rearranged as

\[
\begin{bmatrix}
\Delta e_d - e_d \Delta \delta \\
\Delta e_q + e_d \Delta \delta
\end{bmatrix} =
\begin{bmatrix}
r + \frac{s}{\omega_o} x_d(s) & -x_q(s) \\
x_d(s) & r + \frac{s}{\omega_o} x_q(s)
\end{bmatrix}
\begin{bmatrix}
\Delta i_d - i_d \Delta \delta \\
\Delta i_q + i_d \Delta \delta
\end{bmatrix}
+ \begin{bmatrix}
\frac{s}{\omega_o} [x_q - x_d(s)] i_q - [E_{fd} - (x_d - x_q(s)) i_d] \\
[x_q - x_d(s)] i_q + \frac{s}{\omega_o} [E_{fd} - (x_d - x_q(s)) i_d]
\end{bmatrix}\Delta \delta
\]  \hspace{1cm} (9.108)

Linearizing (9.106), and using (9.102) and (9.103),

\[ \Delta T_e = [x_q - x_d(s)] i_q \Delta i_d + [E_{fd} - (x_d - x_q(s)) i_d] \Delta i_q \]  \hspace{1cm} (9.109)

At shaft torsional frequencies \( \omega_d(s) \) and \( \omega_q(s) \) are approximately equal and each may be assumed to be equal to their average value. At oscillation frequency \( \omega_n \), equations (9.108) and (9.109) may be written as

\[
\begin{bmatrix}
\Delta e_d - e_d \Delta \delta \\
\Delta e_q + e_d \Delta \delta
\end{bmatrix} = -Z \cdot \omega_n \begin{bmatrix}
\Delta i_d - i_d \Delta \delta \\
\Delta i_q + i_d \Delta \delta
\end{bmatrix} + \begin{bmatrix}
a \\
b
\end{bmatrix}
\]  \hspace{1cm} (9.110)

where
\[ Z_G(j\omega_n) = \begin{bmatrix} r + j\frac{\omega_n}{\omega_o} x_d(j\omega_n) & -x_d(j\omega_n) \\ x_d(j\omega_n) & r + j\frac{\omega_n}{\omega_o} x_d(j\omega_n) \end{bmatrix} \]

\[ a = \left[ j\frac{\omega_n}{\omega_o} \left[ x_q - x_d(j\omega_n) \right] i_q - \left[ E_{fd} - (x_d - x_d(j\omega_n))i_d \right] \right] \Delta \delta \]

\[ b = \left[ x_q - x_d(j\omega_n) \right] i_q + j\frac{\omega_n}{\omega_o} \left[ E_{fd} - (x_d - x_d(j\omega_n))i_d \right] \Delta \delta \]

\[ \Delta T_e = [x_q - x_d(j\omega_n)]i_q \Delta i_d + [E_{fd} - (x_d - x_d(j\omega_n))i_d] \Delta i_q \quad (9.111) \]

Note that

\[ Z_G(j\omega_n) = \begin{bmatrix} \frac{1}{2}(Z_{G+} + Z_{G+}) & -j \left( \frac{1}{2}(Z_{G+} - Z_{G+}) \right) \\ j \left( \frac{1}{2}(Z_{G+} - Z_{G+}) \right) & \frac{1}{2}(Z_{G+} + Z_{G+}) \end{bmatrix} \quad (9.112) \]

where

\[ Z_{G+} = r + j\frac{\omega_n + \omega_o}{\omega_o} x_d(j\omega_n) \]

\[ Z_{G-} = r + j\frac{\omega_n - \omega_o}{\omega_o} x_d(-j\omega_n) \]

and

\[ x_d(j\omega_n) = x_d^*(-j\omega_n) \]

From (9.100) and (9.110) we have

\[ \begin{bmatrix} \frac{1}{2}(Z^+ + Z^-) & -j \left( \frac{1}{2}(Z^+ - Z^-) \right) \\ j \left( \frac{1}{2}(Z^+ - Z^-) \right) & \frac{1}{2}(Z^+ + Z^-) \end{bmatrix} \begin{bmatrix} \Delta i_d - i_q \Delta \delta \\ \Delta i_q + i_d \Delta \delta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \]

where

\[ Z^+ = Z_{N+} + Z_{G+}, \quad Z^- = Z_{N-} + Z_{G-} \]

which yields

\[ \begin{bmatrix} \Delta i_d - i_q \Delta \delta \\ \Delta i_q + i_d \Delta \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left( \frac{1}{Z^+} + \frac{1}{Z^-} \right) & j \left( \frac{1}{2} \frac{1}{Z^+} - \frac{1}{2} \frac{1}{Z^-} \right) \\ -j \left( \frac{1}{2} \frac{1}{Z^+} - \frac{1}{2} \frac{1}{Z^-} \right) & \frac{1}{2} \left( \frac{1}{Z^+} + \frac{1}{Z^-} \right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (9.113) \]

Substituting the expressions for \( \Delta i_d \) and \( \Delta i_q \) from (9.113) into (9.111), the air-gap torque can be expressed as
\[ \Delta T_e = K (j \omega_n) \Delta \delta \] (9.114)

The imaginary part of the above expression will be indicative of electrical damping. Note that the damping will depend not only on the system parameters but also on the operating point. The computation of electrical damping from equations (9.111) and (9.113) is quite straightforward once a frequency scan of the system network is obtained. Note that it is not necessary to set \( x_d(s) = x_q(s) \), although this allows some simplification in that the generator and network impedances can be combined.

As a specific example, consider the generator under no-load. At no-load,
\[
i_d = i_q = 0
\]
\[
E_{fd} \approx 1.0
\]
\[
\therefore \quad a = -\Delta \delta
\]
\[
b = j \frac{\omega_n}{\omega_o} \Delta \delta
\]
\[
\Delta T_e = \Delta \delta
\]

From (9.113)
\[
\Delta i_q = -j \frac{1}{2} \left( \frac{1}{Z_+} - \frac{1}{Z_-} \right) (\Delta \delta) + \frac{1}{2} \left( \frac{1}{Z_+} + \frac{1}{Z_-} \right) j \frac{\omega_n}{\omega_o} \Delta \delta
\]

Therefore, the electrical damping is obtained as
\[
D_e = \frac{\omega_o}{\omega_n} \left[ \frac{1}{2} \left( \frac{R_+}{|Z_+|^2} + \frac{\omega_n}{\omega_o} \frac{R_+}{|Z_+|^2} \right) + \frac{1}{2} \left( \frac{-R_-}{|Z_-|^2} + \frac{\omega_n}{\omega_o} \frac{R_-}{|Z_-|^2} \right) \right]
\]

or
\[
D_e = \frac{1}{2} \left( \frac{\omega_o + \omega_n}{\omega_o} \frac{R_+}{|Z_+|^2} - \frac{1}{\omega_o} \frac{\omega_n - \omega_o}{|Z_-|^2} \frac{R_-}{\omega_n} \right) \quad (9.115)
\]

The second term is usually more dominant and therefore the electrical damping due to torsional interaction tends to be negative. Also, when the subsynchronous frequency \( \omega_b - \omega_n \) (the complement of the torsional natural frequency \( \omega_n \)) approaches the network resonant frequency, \( X_- = 0 \). Under these conditions the negative damping can reach a high magnitude, since the resistance is usually small.

Note that if the magnitude of the negative electrical damping exceeds the modal mechanical damping corresponding to that particular frequency, the combined mechanical-electrical system will be unstable.

The method described above can also be used in the analysis of the phenomenon of hunting (the oscillation of the combined turbine-generator system with respect to the power system), which is usually in the 0.05 to 2 Hz range, with or without series capacitor in the system. However, since \( \omega_n \) is small, the approximation \( x_d(j \omega_n) = x_q(j \omega_n) \) cannot be used.
Self-Excitation
Whenever there is a perturbation in a series compensated system, transient subsynchronous currents of both positive and negative sequence will flow in the network. The frequency of the subsynchronous currents will be given by

$$\omega_e = \omega_o \frac{X_C}{X_L}$$

where $X_L$ and $X_C$ are the total equivalent series inductive and capacitive reactances, respectively.

Considering the positive sequence subsynchronous current, since the generator rotor is rotating at or close to synchronous speed, the machine will act as an induction generator feeding energy into the system. Whether or not the subsynchronous current will grow can be determined from the following consideration.

Since $\omega = \omega_e - \omega_o$ is negative, and $T_{do} > T_d'$, $T_{do} > T_d^*$, it is evident that $Z_G$ will have a negative real part the value of which can be readily calculated from equations (9.116) and (9.117). If the magnitude of this negative resistance exceeds the combined resistance of the generator armature and the network, the subsynchronous current will grow, since the decrement factor ($R/2L$) will be negative.

It is evident from equations (9.116) and (9.117) that, the lower the magnitude of $\omega$ (i.e., higher $\omega_o$) the larger the value of the negative real part of $Z_G$. Therefore, the possibility of self-excitation increases at higher series compensation. Similarly, larger values of the time constants will reduce the magnitude of the negative real part and hence lower the possibility of self-excitation. Larger values of $T_d^*$ and $T_{do}^*$ may be obtained by using low resistance pole-face damper windings.

Following the above procedure it can be seen that the real part of $Z_G$, corresponding to the negative sequence subsynchronous current, is positive and therefore this component will always damp out.

Note that self-excitation of purely electrical origin is possible only in the presence of series compensation. For self-excitation to occur the effective system resistance at system natural (subsynchronous) frequency must be negative. Positive resistance, on the other hand, is
TURBINE-GENERATOR SHAFT TORSIONALS

responsible for the negative electrical damping due to torsional interaction (see equation 9.115), which can occur in systems with or without series compensation.

In the state-space approach described earlier both the torsional interaction effect and the self-excitation effect are accounted for simultaneously.

Control of Subsynchronous Oscillation Problems

A number of control measures are available for mitigating the subsynchronous oscillation problems arising from series capacitor compensation of transmission lines. None of these control measures, applied singly, can however be considered to be a complete solution to the problems. Some of these are effective only in specific applications. More than one measure may be needed in any given situation.

From the analysis presented earlier it is evident that the most direct way of eliminating most of the shaft torsional problems would be to provide for large mechanical damping of the shaft system. However, there does not appear to be any practical means of achieving this at the present time.

The theory and principle of operation of some of the control measures will now be described [26 - 30]. Although recent control measures employing power electronics are not included here, the principles of operation are similar.

Static blocking filter

This is a multi-element blocking filter placed in series with the high-voltage winding of the generator step-up transformer at the neutral end. Each element is a high Q parallel LC circuit tuned to block electrical current at a frequency which coincides with the complement of a torsional natural frequency, with negligible increase in impedance to 60 Hz current. The combination of the filter impedance and the transmission impedance introduces parallel resonances at the torsional complementary frequencies and shifts system series resonance points to frequencies that cannot damage the machine.

The static filter provides control of transient torque and torsional interaction effects. Since each filter is tuned to protect an individual unit, the effect of system changes is minimal. The filters’ effectiveness is reduced when detuned due to normal temperature variations, capacitor failure, and changes in system frequency during a disturbance. This may be counteracted by augmenting the natural shaft damping by a supplementary excitation control. This device provides damping by injecting into the voltage regulator of a high initial response excitation system a properly phased sinusoidal signal derived from rotor motion. Also, it may be necessary to increase the basic insulation level of the generator step-up transformer.

Field tests have demonstrated the effectiveness of the static filter.

Line filter

The flow of subsynchronous current at a specific frequency can be blocked by connecting an appropriately sized reactor in parallel with an existing series capacitor. The filter thus formed will have a higher net capacitive reactance at 60 Hz than the original series capacitor and a lower allowable line current. The original capacitive reactance can be restored by connecting additional capacitors in parallel.
If $X_C$ and $X_{Cf}$ are the 60 Hz capacitive reactances of the original series capacitor and the filter capacitor, respectively, and $X_L$ is the 60 Hz inductive reactance of the filter reactor, then

$$X_{Cf} = X_C \left( \frac{60}{f} \right)^2 - 1, \quad X_L = X_C \left( \frac{60}{f} \right)^2 - 1$$

where $f$ is the tuned frequency of the filter.

The 60 Hz MVAR ratings of the filter capacitor and reactor, $Q_{Cf}$ and $Q_L$, are given by

$$Q_{Cf} = \frac{Q_C}{\left( \frac{60}{f} \right)^2 - 1}, \quad \text{and} \quad Q_L = \frac{Q_C}{\left( \frac{60}{f} \right)^2 - 1}$$

where $Q_C$ is the 60 Hz MVAR rating of the original series capacitor bank.

The above expressions show that the MVAR requirements of the filter elements go up rapidly as the tuned frequency increases. The filter would have the best application at low electrical frequencies.

The line filter could be effective in controlling transient torque and torsional interaction problems when a single frequency is involved and the source of the problem is a single line.

**Bypass damping filter**

The filter consists of a resistor in series with a parallel combination of a reactor and capacitor tuned to the system frequency (60 Hz). When connected across the series capacitor, the filter provides an inductive/resistive bypass path for the flow of subsynchronous currents while blocking the normal frequency current. By introducing positive resistances in the circuit for subsynchronous currents the filter can be effective in controlling electrical self-excitation (the induction generator effect).

When a single line and a single torsional frequency is involved, the bypass damping filter can be used in conjunction with the line filter to counter torsional interaction as well as self-excitation.

**Pole-face damper windings**

Pole-face damper windings can be effective in controlling electrical self-excitation (the induction generator effect).

Self-excitation results from a negative net system resistance at series resonance. Since the only source of negative resistance is the generator, an effective solution would be to reduce the negative resistance contribution of the generator at subsynchronous frequencies. This can be achieved by adding pole-face damper windings to the generators.

In general, it is not practical to install pole-face damper windings on existing machines. However, it is relatively inexpensive to install them on new machines.

Note that pole-face damper windings would not be effective in controlling torsional interaction and transient torque problems.
Dynamic filter

The dynamic filter is an active device placed in series with the generator to cancel the subsynchronous voltage generated by torsional oscillations. A signal derived from rotor motion is used to produce a voltage in phase opposition and of sufficient magnitude to counter the subsynchronous voltage generated by rotor oscillation.

The dynamic filter would be effective in controlling torsional interaction problems. However, it would do little to alleviate transient torque problems because of the inherent long time constants of the control system.

Dynamic stabilizer

The dynamic stabilizer is a thyristor modulated shunt inductive load connected to the terminals of the generator. The modulation of the reactance is determined by control signals derived from rotor speed deviation.

The dynamic stabilizer can be visualized as a high-impedance inductive load where the inductive susceptance is being sinusoidally modulated at the rotor oscillation frequency \( \omega_n \). For an applied voltage at system frequency \( \omega_0 \), the modulation will produce currents through the reactor at frequencies \( \omega_0 \pm \omega_n \). The reactor currents will be divided between the generator and the transmission network.

If \( \Delta i \) is the current through the reactor at the subsynchronous frequency, the portion of this current through the generator, considering an equivalent generator transmission system, is

\[
\Delta i_G = \frac{R_N + j(X_N - X_C)}{R_G + R_N + j(X_G + X_N - X_C)} \Delta i
\]

where the subscripts \( G \) and \( N \) refer to the generator and network quantities respectively, and \( X_C \) is the series capacitor reactance.

At electrical resonance

\[
X_G + X_N - X_C = 0
\]

\[
\therefore \Delta i_G = \frac{R_N - jX_G}{R_G + R_N} \Delta i \approx \frac{-jX_G}{R_G + R_N} \Delta i
\]

Thus the generator current is amplified in inverse proportion to the net system resistance. This is the same amplification as that for the current resulting from the voltage produced by rotor oscillation. Therefore, with proper gain and phase control, the two currents will cancel resulting in no net oscillating torque.

The supersynchronous component of the reactor current will not be amplified by electrical resonance and therefore is not a primary consideration in determining the control strategy.

Since the stabilizer responds to the electrical network in the same manner as the torsional interaction, the stabilizer control will be independent of network configuration.

The dynamic stabilizer does not provide protection against self-excitation and transient torque problems although it can provide damping for rotor oscillations.
Also, there must be provisions for minimizing harmonic currents injected into the system by the stabilizer. Harmonics can be controlled by the addition of a special multi-winding transformer and by delta connection of the stabilizer reactors.

**Control of series capacitors**

The source of most subsynchronous oscillation problems is the series capacitor in the transmission system. Control of series capacitors may therefore be used to mitigate some of these problems.

Series capacitors are protected by limiting the voltage across them under abnormal conditions. The usual protection for a capacitor bank is provided by a gap spark-over scheme that short circuits the capacitors when the voltage reaches a predetermined level. The scheme usually consists of one or more pilot or trigger gap, and a main or power gap that takes the brunt of the discharge. The spark-over is initiated by the trigger gap which causes spark-over of the main gap. The spark-over is accomplished within a few micro-seconds from the time the spark-over voltage level is reached.

The spark-over voltage is usually adjustable from 2.0 to 3.5 times the peak voltage across the capacitor corresponding to rated continuous current. A bypass switch is included to provide a means of diverting the current from the capacitor equipment.

The most common cause of capacitor gap flashing is a system fault. When this occurs, it is important, from a stability standpoint, to have the capacitors in the unfa ulted lines reinserted within a few cycles after fault clearing. Reinserter produces a transient voltage across the capacitors. In some situations, because of high post reinsertion voltages or swing voltages, the capacitor gap, even at full setting, may not be adequate to insure consistent reinsertion. In some installations successful reinsertion performance has been achieved by switching non-linear resistors in parallel with the capacitors following gap flashing and then removing them a few cycles after reinsertion. The presence of parallel resistors greatly reduces the voltage across the capacitors during reinsertion.

Relays are provided to detect extended arcing of the gaps and cause the bypass switch to close and divert current from these gaps. Manual opening of the bypass switches is required for insertion of the capacitors.

**Reduced series capacitor gap setting**

A reduced series capacitor gap setting directly results in a reduced transient shaft torque. The reduction in torque is realized due to a reduction in the level of short circuit currents and hence the maximum energy stored in the capacitors during the fault. A lower gap setting may, however, have the adverse effect of lowering the probability of successful reinsertion following fault clearance. The probability of successful reinsertion may be improved by adding transient voltage limiter resistors.

**Dual-gap protection scheme**

An effective means of reducing the transient torques while maintaining the capacitor reinsertion capability is to equip the series capacitors with a dual-gap protection scheme. In this scheme, a lower-level gap is provided for normal operation to limit the transient fault current. Spark-over of the lower-level gap causes spark-over of a higher-level gap which in turn causes spark-over of the main power gap and opening of a switch provided in series with the lower gap.
The high-level gap permits the capacitor to be reinserted against the transient reinsertion voltage and remain reinserted during system swings after the fault has been cleared. The low-level gap is then restored after a time delay of several seconds to allow the system swing currents to decay sufficiently and avoid a restrike.

Some risk of shaft damage remains, however. For example, if the system swings are large following fault clearance or another fault occurs immediately after reinsertion but before the lower-gap has been restored, the shafts may be exposed to damaging torques.

**Extended-range dual-gap scheme**

The minimum setting of the low-level gap in the dual-gap scheme is about 65% of the high-level gap setting. In the extended-range scheme the low-level setting is reduced to about 1.5 times the rated capacitor voltage, while the high-level setting is maintained at 3.5 times the rated voltage. The concept of the extended-range scheme is otherwise same as that of the dual-gap scheme. The lowering of the low-level gap setting is made possible by the application of a pulse transformer to provide the additional voltage required to force gap flashover. Using the extended-range scheme further reduction of transient shaft torques is possible.

**Forced gap flashing**

The capacitor gap flashing by a remote signal has been proposed as a solution to the transient torque problem. The remote signal would be derived from system sensing and logic circuitry and gap flashing would be initiated for predetermined system conditions and fault locations. Capacitor insertion would be automatic upon removal of the bypass signal.

**Fast insertion of series capacitor**

In this scheme the level of series compensation would be reduced to a level that would eliminate subsynchronous oscillation problems but would still allow acceptable steady state operation. Additional series capacitors would be temporarily inserted into the system during a disturbance to enhance transient stability.

The scheme is not workable in situations where high levels of compensation are necessary for normal steady state operation. Also, the requirement of capacitor insertion time for transient torque reduction and transient stability enhancement can be conflicting.

**Coordinated series capacitor use with loading**

Turbogenerator modal mechanical damping tends to increase with unit loading. In some situations the only subsynchronous oscillation problem that is likely to occur is torsional interaction when the generator unit loading is below a minimum level. In such cases the series capacitors may be bypassed as the unit loading drops below this minimum level.

**Non-linear bypass resistors**

Series capacitor protection schemes employing non-linear resistors offer effective means of mitigating transient torque problems. In one such scheme the capacitor overvoltage is limited by inserting a non-linear silicon carbide resistor in parallel with the series capacitor by a spark-over gap when the voltage across the capacitor reaches a predetermined level. This provides a parallel path for capacitor discharge after fault clearing until the gap arc is extinguished and the resistor is removed. This limits the current discharge through the generator.
Another scheme employs highly non-linear Zinc oxide (ZnO) resistors that provide passive overvoltage protection during faults. A ZnO resistor acts as a voltage clipper. The resistor is permanently connected across the series capacitor bank and under normal operating conditions conducts only a small leakage current. During a fault the high voltage developed across the capacitor causes the resistor to conduct thereby limiting the capacitor voltage. By employing resistors of appropriate clipping level the transient torque problem can be mitigated.

The ZnO resistors have a short-time energy absorption limit. The capacitor bank is bypassed when this limit is exceeded.

**Intermittent bypassing of series capacitors**

The presence of subsynchronous voltages (and currents) in a 60 Hz system will cause the periods of each half cycle of the voltage (and current) waveform to be unequal. The theory behind the scheme is that the subsynchronous components can be removed or rendered ineffective by momentarily discharging the capacitor through a resistor whenever the half cycle period of the voltage across it exceeds the normal frequency half cycle period of 8.33 ms.

In the basic scheme a linear resistor in series with thyristors arranged in reverse parallel is connected across the series capacitor. The half cycle periods are timed from the zero voltage crossing points. Whenever the duration of the half cycle exceeds 8.33 ms, the corresponding thyristor is fired and the capacitor is discharged through the resistor, bringing about the current zero sooner than it would be otherwise. Thyristors stop conducting when the capacitor voltage (and therefore the thyristor current) drops to zero. Half cycle measurement restarts at a new voltage zero. No thyristor is fired for half cycles shorter than 8.33 ms.

The operation of the scheme is independent for each phase and does not require any signal from the generator or any other specific subsynchronous signals.

The scheme appears to be promising, although in actual application some complications are possible.

**Protective relays**

Several relays are available for protecting turbine-generators against subsynchronous oscillation problems. In these relays turbine-generator shaft torsional status is determined either by measuring the generator electrical quantities or by measuring the shaft system mechanical response. The relays operate to disconnect the machines from the system whenever excessive shaft stresses are indicated.

These relays are generally too slow to provide effective transient torque protection. Note that unit tripping must occur well before the shaft torques reach peak levels. This is because tripping itself produces large shaft oscillations and torques which when superimposed on the existing oscillations may produce more damage than if the unit was not tripped.

The relays can be used as back-up to one or more of the primary protections that may be applied to the generator unit.

**Torsional motion relay**

This relay responds by detecting excessive turbine-generator shaft torsional stresses from measurements of shaft speed deviation. Speed sensors in the form of magnetic pick-ups are placed in proximity to toothed wheels at each end of the turbine-generator shafts. For steady
shaft speed, the pick-up coil voltage is a constant amplitude sinusoid. The presence of torsional oscillations produces a frequency modulation of the constant amplitude voltage at the coil. The coil voltage is processed through a differential speed transducer to produce a voltage proportional to shaft speed deviation. Bandpass filters tuned to the machine's natural frequencies are used for a modal decomposition of the speed deviation. The modal speed deviations measured at any one location are relatable to shaft torques through the shaft system mode shapes. The filter outputs act as input to a relay tripping logic device which provides the trip signal.

**Subsynchronous over-current relay**

This is a modified negative sequence current relay with pass and block filter added. The relay detects positive sequence subsynchronous currents in the 20-40 Hz range. The device is equipped with two current level detectors which are separately adjustable. The minimum pick-up level is about 3% of the rated generator current. The relay is intended to provide protection from electrical self-excitation and torsional interaction by initiating generator tripping for sustained subsynchronous oscillations.

Sufficient time delay is required to override subsynchronous currents produced by non-critical faults and switching to avoid unnecessary trips. The device cannot therefore provide transient torque protection.

**Subsynchronous oscillation relay**

This relay also works by responding to subsynchronous currents in the generator armature. However, the detection of subsynchronous current and relay trip logic are more elaborate.

From the input current signal the subsynchronous components are extracted by using a combined modulation and filtering scheme. A multiplier circuit performs a synchronous modulation by multiplying the currents with a synchronous frequency voltage. The synchronous component of the input signal is modulated into a dc component and a double frequency component. Each positive sequence subsynchronous component of frequency \( f_e \) is modulated into components of frequencies \( f_o \pm f_e \). A wideband filter (15 - 45 Hz) extracts the subsynchronous components. The subsynchronous signal, which is at frequency \( f_o - f_e \), is directly related to both the magnitude and frequency of the electrical torque and the resulting mechanical stresses in the shafts.

The relay can detect subsynchronous signals that are either growing or decaying with time. Separate modules utilizing special logics are provided to detect excessive transient torque, torsional interaction and electrical self-excitation. The modules analyze the signals independently and initiate a trip whenever a trip criterion is satisfied.

**Generator tripping**

Generator tripping for predetermined system conditions and fault locations can be used as protection against subsynchronous oscillation problems. This solution requires that all system conditions and fault locations that would result in subsynchronous problems be known from prior studies. System sensing and logic circuitry can then be employed to initiate tripping fast enough to limit shaft torques to acceptable levels.

**Use of shunt compensation**

Shunt compensation offers an effective means of increasing the transmission capability [31-32]. It can be shown that for transmission distances and shunt compensation levels that would normally be encountered, the system natural frequencies are all super-synchronous. Therefore,
the subsynchronous oscillation problems associated with series compensation are eliminated. Although possibility of resonance at a super-synchronous frequency exists, its impact is not likely to be serious due to the large positive damping of the oscillations.

By applying shunt compensation at intermediate points along a transmission line, the power transmission limit can theoretically be raised to almost any level. Two major problems of increasing power transfer by shunt compensation using fixed capacitors are: (1) excessive voltage at capacitor application points at light loads, and (2) possibility of voltage collapse under certain operating conditions. These can be effectively handled by continuous voltage control employing thyristor controlled static VAR systems.

The voltage stability problem associated with increased power transfer using shunt compensation and its solution are discussed in detail in Chapter 10.

It can be shown that for a given increase in power transmission over a single transmission line, the total reactive requirements and the total costs, installed, for series and shunt compensation are similar. The comparison does not, however, include the cost of the control and protective equipment that might be needed if the series compensated system exhibits subsynchronous oscillation problems. Additionally, since the shunt compensation is connected to the buses, a single capacitor bank can serve more than one line or line section connected to the same bus, and continue serving the remaining lines or line section when one of them is outaged. This leads to economy.

An important difference between the operating characteristics of the two types of compensation is that, for the same power transmission, the voltage angle difference across the transmission line is much larger for shunt compensation than for series compensation. The use of shunt compensation may therefore cause some difficulty in line loading in some situations by adversely affecting the flow patterns on parallel paths. Note, however, that series compensation is also known to cause line loading difficulty and contribute to circulating power flow.

If an excessive increase in line angle is unacceptable, then a combination of series and shunt compensation can be used to permit increased power transmission around a desired line angle. This may allow the series compensation to be reduced to a level where subsynchronous oscillation will no longer be a problem. If so, the shunt compensation may be regarded as a control measure for subsynchronous oscillation problems.

References

9-43


CHAPTER 10
VOLTAGE STABILITY

As given in Chapter 2, voltage stability, assuming that the stability in question involves voltage only, can be defined as:

A power system at a given operating state and subject to a given disturbance is voltage stable if voltages approach post-disturbance equilibrium values.

In simple terms the stability conditions may be stated as follows: Assuming that an equilibrium satisfying the operating constraints, i.e., a viable power flow solution, exists, the equilibrium state is small disturbance stable if all the eigenvalues of the system, linearized around the operating point, have negative real parts. Large disturbance stability is assured if the system state at the end of the disturbance lies within the region of attraction of the stable equilibrium point of the post-disturbance system. Although the above conditions apply to any type of stability, in this chapter we will be concerned only with voltage stability.

Uncontrollable increase in voltage caused by self excitation has been discussed in Chapters 5 and 9. In this chapter we will be concerned with uncontrollable voltage decrease resulting in voltage collapse. Voltage instability can be local or system-wide. If local, the rest of the system need not feel the impact of the instability if the affected area is promptly disconnected from the rest of the system by protective action.

Voltage collapse can occur in the absence of voltage instability, if we define voltage collapse as drop in voltage below a point where the performance of the connected devices will not meet design criteria. For example, consider a system serving only heating and/or lighting loads. As we will see later, for such loads voltage instability cannot occur as per our definition of voltage stability. However, as more and more load is added the voltage will keep dropping. If the loading continues there will come a point beyond which the demanded MW will not be served, and the voltage will be too low for the devices to produce sufficient heat or light. For all practical purposes we can say that the voltage has collapsed.

Angle instability can also lead to voltage collapse which might appear as voltage instability. Consider power being transmitted between two points over a transmission line. As the power transmitted approaches the angle stability limit (the angle across the line reaching 90°), the voltage at the mid-section of the line will be depressed and any load connected there will experience voltage collapse. The same situation may arise during power swings across transmission lines following a disturbance.

Before we undertake voltage stability analyses considering all the relevant dynamics, it would be useful, as in the case of angle (synchronous) stability, to start with the basics of power flow and power limits, considering voltage as the main issue. In the discussion of power limits in connection with angle stability, voltages were assumed to be either constant (generator internal voltages) or controlled so that we could be concerned only with the angle stability issues. In this chapter we will be concerned mostly with voltages at the load delivery point where the voltages are under limited or no control.
Steady-state Analysis of the Voltage-Reactive Power Problem

Two bus systems

The power transmitted between two points in a transmission system is approximately given by the expression

\[ P = \frac{E_1 E_2}{X} \sin \delta \]  

(10.1)

where \( E_1 \) and \( E_2 \) are the voltages at the two points, \( \delta \) is the angle difference and \( X \) is the reactance between them.

If the voltages are maintained constant, the maximum power transmission occurs when \( \delta = 90^\circ \), and

\[ P_{\text{max}} = \frac{E_1 E_2}{X} \]  

(10.2)

However, the condition of constant voltages is satisfied only in specific situations, for example, when the voltages \( E_1 \) and \( E_2 \) represent the fixed internal voltages of two machines (a generator and an equivalent motor representing the load) at the ends of the transmission line. More often, one or both voltages will vary as the power transmitted changes. It will be seen that for a given sending-end voltage, \( V_S \), and in the absence of specific voltage support at the receiving end, the maximum power transfer at unity power factor occurs when the angle difference between the sending- and receiving-end voltages is equal to 45°. At this point the receiving-end voltage is equal to 0.707 times the sending-end voltage and the power transfer is

\[ P_{\text{max}} = \frac{V_S^2}{2X} \]  

(10.3)

Relationship between power flow and voltage drop

Consider the simple system shown in Figure 10.1. The sending-end voltage \( V_S \) is held constant. For simplicity, line charging has been neglected.

![Fig. 10.1 A simple two-bus transmission system](image)

At the receiving end

\[ V_R I^* = P + jQ \]

\[ \therefore \quad I = \frac{P - jQ}{V_R} \]

using \( V_R \) as reference.

\[ \therefore \quad V_S = V_R + \left( \frac{P - jQ}{V_R} \right)(R + jX) \]
Approximate relationships

Since in a transmission line the resistance is small in relation to the reactance \( R \ll X \), it can be neglected without affecting the accuracy significantly. In the analysis presented here, inclusion of resistance will only serve to obscure the more important issues. The effect of line resistance will therefore be ignored in all the subsequent analyses.

The approximate relationship of equation (10.4), neglecting resistance, is shown in the phasor diagram of Figure 10.2.

\[
V_S = \left( V_R + \frac{PR}{V_R} + \frac{QX}{V_R} \right) + j \left( \frac{PX}{V_R} - \frac{QR}{V_R} \right) \tag{10.4}
\]

Since, at normal levels of power flow

\[
\frac{PX}{V_R} \ll V_R + \frac{QX}{V_R}
\]

\[
|V_S| \approx V_R + \frac{QX}{V_R} \tag{10.5}
\]

(Note: If \( P \) and \( Q \) are specified at the sending end, \( |V_R| \approx V_S - \frac{QX}{V_S} \))

Equation (10.5) shows that the voltage drop at the receiving end is predominantly dependent on the reactive power delivered or the power factor of the load. At nominal levels of power transfer at unity power factor over short to medium length transmission lines, the voltage drop at the receiving end is minimal.

From (10.5)

\[
\frac{dV_R}{dQ} \approx \frac{X}{V_S - 2V_R} \tag{10.6}
\]

Under normal operating conditions

\[
V_S \approx V_R \approx 1.0
\]

\[
\therefore \frac{dV_R}{dQ} \approx -X \tag{10.7}
\]

\( = -(\text{reciprocal of the short-circuit current}) \)
VOLTAGE STABILITY

As the reactive power flow increases and/or the active power reaches excessively high level, $V_R$ drops well below nominal value and the magnitude of $\frac{dV_R}{dQ}$ increases significantly.

At low levels of active power flow, as

$$Q \rightarrow \frac{V_S^2}{4X}, \quad V_R \rightarrow \frac{V_S}{2}, \quad \text{and} \quad \frac{dV_R}{dQ} \rightarrow -\infty$$

At $P = 0$, $\frac{dV_R}{dQ} = -\infty$ when

$$Q = \frac{V_S^2}{4X}$$

This is the maximum amount of reactive power that can be delivered at the receiving end. At this reactive level

$$V_R = \frac{V_S}{2}$$

$V_R - V_S$ relationships under various receiving-end power factor conditions are illustrated by the phasor diagrams in Figure 10.3.

Fig. 10.3 Phasor diagrams showing the $V_R - V_S$ relationships at various power factors.
From the above phasor diagrams it is clear that at a given power transfer, in order to maintain the receiving-end voltage magnitude close to that at the sending end, a certain amount of reactive support at the receiving end is necessary.

**Power limit without specific voltage support at the receiving end**

Consider the system shown in Figure 10.1, where the complex power \( P + jQ \) is being delivered at the receiving end. From equation (10.4), neglecting line resistance,

\[
V^2_s = \left( V + \frac{QX}{V} \right)^2 + \left( \frac{PX}{V} \right)^2
\]

from which

\[
V^2_r = \frac{(V^2_s - 2QX) \pm \sqrt{(V^2_s - 2QX)^2 - 4(P^2X^2 + Q^2X^2)}}{2} \quad (10.10)
\]

Equation (10.10) shows that when the expression under the radical sign is positive, two solutions for the receiving-end voltage are possible, both physically realizable. The solution having the larger value (corresponding to the positive sign before the radical) represents the normal operating condition. The solution with the lower value results in a correspondingly higher current in order to meet the power demand. (Note that at \( P = 0, Q = 0 \), the two solutions correspond to open- and short-circuit conditions, respectively.) The second solution is physically realizable only for certain load types or load-voltage control combinations. In this operating state, an increase in the power demand would be accompanied by an increase in current with a disproportionately large drop in voltage and a runaway situation would occur if the load attempted to restore to the demanded value. A total voltage collapse would possibly be averted since below a certain voltage level most loads will cease to act as constant power load and resort to constant current or constant impedance behavior. Operation in this region is not desirable. As shown later, stable operation for constant power load can be realized in this operating state by employing special controls. However, this may be impractical in most situations.

Writing \( Q = P \tan \phi \), where \( \phi \) is the power factor angle, (10.10) can also be expressed as

\[
V^2_r = \frac{(V^2_s - 2PX \tan \phi) \pm \sqrt{(V^2_s - 2PX \tan \phi)^2 - 4P^2X^2(1 + \tan^2 \phi)}}{2} \quad (10.11)
\]

As the expression under the radical sign approaches zero, the two solutions approach each other. No solution exists when the expression becomes negative. The maximum power that can be delivered at the receiving end at a given power factor is, therefore, obtained by equating the expression under the radical sign to zero. The maximum or critical power thus obtained is given by

\[
P_{\text{max}} = \frac{V^2_s}{2X} \frac{1 - \sin \phi}{\cos \phi} \quad (10.12)
\]

The receiving-end voltage at the maximum power delivered is given by

\[
V_r = \frac{V_s}{\sqrt{2}} \sqrt{\frac{1 - \sin \phi}{\cos^2 \phi}} \quad (10.13)
\]
Note that at unity power factor, $\phi = 0$. Therefore

$$P_{\text{max}} = \frac{V_s^2}{2X}$$  \hspace{1cm} (10.14)

and

$$V_r = \frac{V_s}{\sqrt{2}} \approx 0.7V_s$$  \hspace{1cm} (18.15)

Using the expression for power, $P = \frac{V_sV_r}{X} \sin \delta$, and equations (10.14) and (10.15), $\delta$ at maximum or critical power is given by $\delta = 45^\circ$. In the absence of specific voltage support at the receiving end, the maximum power that can be delivered at unity power factor is, therefore, given by equation (10.14), and this occurs at $\delta = 45^\circ$.

At power factors other than unity, from equations (10.12) and (10.13), and using the usual expression for power transfer,

$$\frac{V_s^2}{2X} \frac{1 - \sin \phi}{\cos \phi} = \frac{V_rV_s}{X} \sin \delta = \frac{V_s^2}{\sqrt{2X}} \sqrt{\frac{1 - \sin \phi}{\cos^2 \phi}} \sin \delta$$

from which

$$\sin \delta = \sqrt{\frac{1 - \sin \phi}{2}}$$  \hspace{1cm} (10.16)

Therefore, at lagging power factors, the maximum power occurs at a power angle less than $45^\circ$. This point has important bearings in actual power system operation.

**Problem:** Show that when the transmission line has appreciable resistance, the maximum power transfer at unity $\text{pf}$ in the absence of specific voltage support at the receiving end occurs at a power angle $\delta = \alpha/2$, where $\alpha$ is given by $\alpha = \tan^{-1}(X/R)$, where $R$ is the line resistance.

Plots of receiving-end voltage against power delivered at various power factors for a typical system are shown in Figure 10.4. For satisfactory performance with good voltage profile the operation should normally be restricted on the upper portion of the curves, well away from the loading limit. It can be seen that when the system is operating close to the loading limit at a given power factor, a slight change in the power factor can initiate a run away situation if the load tended to restore to the demanded level. It can also be seen that although at unity and lagging power factors the receiving-end voltage continuously drops as the power delivered is increased and the voltage at maximum power is quite low compared with the sending-end voltage, the situation is different at leading power factors. At leading power factors, the receiving-end voltage can actually rise with increase in the power delivered, and the voltage at maximum power can be quite high. In other words, at leading power factors, or if sufficient reactive support is provided at the receiving end, the voltage at the maximum power point can be comparable to or even higher than the sending-end voltage, although the maximum power would be quite high. Therefore, the magnitude of the receiving-end voltage alone is not a good indicator of closeness to the critical or maximum power point. In the same way, the change in the receiving-end voltage corresponding to a change in the power delivered, $dV_r/dP$, cannot be used as a reliable guide to predict closeness to the maximum power point.
The change in the receiving-end voltage with change in the reactive power at the receiving end, on the other hand, is more or less independent of the power factor of the load, as seen from the plots shown in Figure 10.4. This conclusion also follows from the analysis presented earlier, where the observation was made that the receiving-end voltage is much more sensitive to reactive power than to real power.

Fig. 10.4 Typical receiving-end voltage characteristics.

From (10.10) and (10.11), one can obtain the following derivatives:

\[
\left( \frac{dV_R}{dP} \right)_{\text{constant } Q} = \pm \frac{-PX^2}{V_R \sqrt{(V_S^2 - 2QX)^2 - 4(P^2X^2 + Q^2X^2)}} \quad (10.17)
\]

\[
\left( \frac{dV_R}{dP} \right)_{\text{constant } \phi} = \frac{-X \tan \phi}{2V_R} \left[ 1 \pm \frac{V_S^2}{\sqrt{(V_S^2 - 2PX \tan \phi)^2 - 4P^2X^2(1 + \tan^2 \phi)}} \right] \quad (10.18)
\]

\[
\left( \frac{dV_R}{dQ} \right)_{\text{constant } P} = \frac{-X}{2V_R} \left[ 1 \pm \frac{V_S^2}{\sqrt{(V_S^2 - 2QX)^2 - 4(P^2X^2 + Q^2X^2)}} \right] \quad (10.19)
\]

and

\[
\left( \frac{dV_R}{d\phi} \right)_{\text{constant } P} = \frac{-PX \sec^2 \phi}{2V_R} \left[ 1 \pm \frac{V_S^2}{\sqrt{(V_S^2 - 2PX \tan \phi)^2 - 4P^2X^2(1 + \tan^2 \phi)}} \right] \quad (10.20)
\]

All these derivatives approach infinity as the maximum power point is approached, since at that point the expressions under the radical signs are zero. Also, at the first solution point, corresponding to the positive sign before the radical in equation (10.10) or (10.11), the derivatives are negative except for the one given by equation (10.18), which may be positive or
negative depending on the power factor (leading or lagging) and the power level. At the second solution point the derivatives are positive.

From the above expressions, it is also clear that for normal transmission line parameters, \( \frac{dV_R}{dQ} \) (or \( \frac{dV_R}{d\phi} \)) has greater magnitude than \( \frac{dV_R}{dP} \) at any point in the upper portion of the curves in Figure 10.4, and that it steadily increases as the maximum power point is approached. Although at the maximum power point both \( \frac{dV_R}{dP} \) and \( \frac{dV_R}{dQ} \) become infinite, prior to reaching the critical point \( \frac{dV_R}{dP} \) may behave anomalously; at least, its variation with change in power level near the critical point may not be indicative of the proximity to the critical point.

\( \frac{dV_R}{dQ} \), on the other hand, has the desirable property of gradually increasing in magnitude as the critical point is approached, independent of the power factor of the load. As shown earlier, also from (10.19), \( \frac{dV_R}{dQ} \approx -X \) (reciprocal of the short-circuit level at the receiving-end bus) in the normal operating range. Any significant departure from this value would be indicative of undesirable situation.

From equations (10.10) or (10.11) curves showing \( V_R \) against \( V_S \) could be drawn for given \( P \) and \( Q \) (or pf) as shown in Figure 10.5. For each value of \( V_S \), two values of \( V_R \) are possible corresponding to the two signs before the radical. At the maximum power point these two values coincide and \( \frac{dV_R}{dV_S} \) is infinite. For values of \( V_S \) lower than this no solution exists. \( \frac{dV_R}{dV_S} \) is positive at values of \( V_R \) corresponding to the positive sign before the radical and represents operation in the upper portion of the curve. Similarly, a negative value of \( \frac{dV_R}{dV_S} \) signifies operation in the lower portion as indicated in Figure 10.5.

![Fig. 10.5 Plots of \( V_R \) against \( V_S \) for given \( P \) and \( Q \).](image)

From (10.10), one can derive \( \frac{dV_R}{dV_S} \) as

\[
\left( \frac{dV_R}{dV_S} \right) = \frac{V_S}{2V_R} \left[ 1 \pm \frac{V_S^2 - 2QX}{\sqrt{(V_S^2 - 2QX)^2 - 4(P^2 X^2 + Q^2 X^2)}} \right]
\]

Equation (10.21) shows that \( \frac{dV_R}{dV_S} \) is positive in the normal operating region. Its value is normally close to unity for operation well within the maximum power point. It rapidly increases as the critical point is approached, becoming infinity at the critical point. A negative value would indicate operation in the lower portion of the curve.
Although the criterion $dV_R/dV_S$ is equivalent to the criterion $dV_R/dQ$ and as convenient to apply in the simple system studied so far, its application in more complex network can be cumbersome. The use of other criteria, for example, $dQ_S/dQ_R$, is also possible. However, these are not deemed suitable for general application.

**Power transfer at constant sending and receiving-end voltage**

At constant sending- and receiving-end voltages, the power transfer increases with $\delta$ (equation 10.1)) until $\delta$ reaches 90°. For a given sending-end voltage, the receiving-end voltage can be held constant by appropriately adjusting the reactive support at the receiving end as the power transfer changes. The required reactive support to maintain the receiving-end voltage at a particular level can be easily determined from the circle diagrams shown in Figure 10.6 (see Chapter 1 for details of circle diagram). In Figure 10.6, a series of circles have been drawn corresponding to a range of values for the receiving-end voltage, for a given sending-end voltage. For a given power transfer, the required reactive support to maintain the receiving-end voltage at a specified level can be read off the circle corresponding to that voltage. From Figure 10.6, it is also possible to determine the receiving-end voltage for a given power and power factor by drawing the constant power factor lines and noting the coordinate of the intersections of these lines with the circles. In this way one can graphically construct the curves shown in Figure 10.4.

![Figure 10.6 Receiving-end circle diagram, $V_S = 1.0$.](image)

However, it will become clear from the analysis presented later in the chapter that, although with reactive support at the receiving-end power transfer will increase with increasing $\delta$, until $\delta$ reaches 90°, in the absence of continuous voltage control, operation at $\delta > 45°$ (at unity power factor) is not desirable.
**Effect of load characteristics**

**Constant current load**

Consider a system supplying a constant current load $I_p - jI_q$ as shown in Figure 10.7.

![Fig. 10.7 Schematic of a system supplying a constant current load and phasor diagram.](image)

From Figure 10.7 we can write, using $V_R$ as reference,

$$V_S \angle \delta = V_R + (I_p - jI_q) jX$$

from which

$$V_S^2 = (V_R + I_qX)^2 + I_p^2 X^2$$

The physically realizable solution for $V_R$ is obtained as

$$V_R = \sqrt{V_S^2 - I_p^2 X^2} - I_qX$$  \hspace{1cm} (10.22)

A necessary condition for the solution to exist is

$$V_S \geq I_p X \quad \text{or} \quad I_p \leq V_S / X \quad \text{-- the short-circuit current at the receiving end.}$$

However, for a non-zero solution of $V_R$ at $I_p = V_S / X$, the load must have a leading power factor ($I_q$ negative).

From (10.22) we can obtain the following partial derivatives:

$$\frac{\partial V_R}{\partial I_p} = -\frac{I_p X}{\sqrt{V_S^2 - I_p^2 X^2}} \hspace{1cm} \frac{\partial V_R}{\partial I_q} = -X \hspace{1cm} \frac{\partial V_R}{\partial V_S} = \frac{V_S}{\sqrt{V_S^2 - I_p^2 X^2}}$$

The first two derivatives are always negative and the third is always positive, indicating that the current delivered can increase continuously, up to the limiting value (the short-circuit current), although beyond a certain point the demanded MW/MVA will not be served.

Note that for $I_q = 0$, $V_R = \sqrt{V_S^2 - I_p^2 X^2}$

Therefore, the real power delivered

$$P = I_p V_R = I_p \sqrt{V_S^2 - I_p^2 X^2}$$

Taking the partial derivative with respect to $I_p$ and equating to zero, we obtain the condition for maximum power that can be delivered as
\[ I_p = \frac{V_S}{\sqrt{2}X} \]

and
\[ V_R = \sqrt{V_S^2 - I_p^2X^2} = \frac{V_S}{\sqrt{2}} \]

from which
\[ P_{\text{max}} = \frac{V_S^2}{2X} \]

\( \delta \) at maximum power is therefore equal to 45°.

The conditions for maximum power are thus the same as in constant power load.

**Constant impedance load**

Consider an impedance load \( R_L + jX_L \), or in terms of admittance, \( G_L - jB_L \), where
\[ G_L = \frac{R_L}{R_L^2 + X_L^2}, \quad \text{and} \quad B_L = \frac{X_L}{R_L^2 + X_L^2} \]

We can write
\[ V_S \angle \delta = V_R + V_R (G_L - jB_L) jX \]

from which
\[ V_R = \frac{V_S}{\sqrt{(1 + B_L X)^2 + G_L^2 X^2}} \quad (10.23) \]

The above expression shows that for constant impedance load the solution for \( V_R \) is unique.

Note that at unity power factor, \( B_L = 0 \), and
\[ V_R = \frac{V_S}{\sqrt{1 + G_L^2 X^2}} \]

\[ \therefore P = V_R^2 G_L = \frac{V_S^2 G_L}{1 + G_L^2 X^2} \]

Taking the partial derivative with respect to \( G_L \) and equating to zero, we obtain the condition for maximum power as
\[ G_L X = 1 \]

\[ \therefore P_{\text{max}} = \frac{V_S^2}{2X}, \quad V_R = \frac{V_S}{\sqrt{2}}, \quad \text{and} \quad \delta_{\text{max}} = 45° \]

**Effect of reactive support at the receiving end by shunt capacitors**

Consider the system shown in Figure 10.8 where reactive support is provided at the receiving end by shunt capacitors of admittance \( B = \omega C \).
The system of Figure 10.8 can be converted to the Thevenin equivalent shown in Figure 10.9, where the equivalent sending-end voltage and the equivalent line reactance are as shown in the figure.

\[ V_S' = \frac{V_B}{1 - XB} \frac{jX - jX}{1 - XB} \]

Fig. 10.9 Thevenin equivalent of system of Fig. 10.8.

The system of Figure 10.9 is identical to the system studied earlier. Therefore, the expressions derived earlier apply provided \( V_S \) and \( X \) are replaced by their equivalent values.

For example, at unity power factor \((Q = 0)\), the receiving-end voltage is given by, from equation (10.10)

\[
V_R^2 = \frac{\left( \frac{V_S}{1 - XB} \right)^2 \pm \sqrt{\left( \frac{V_S}{1 - XB} \right)^4 - 4P^2 \frac{X}{1 - XB}}}{2}
\]  \hspace{1cm} (10.24)

At maximum power

\[
V_R = \frac{V_S}{\sqrt{2(1 - XB)}} \approx \frac{0.7V_S}{1 - XB}
\]  \hspace{1cm} (10.25)

and

\[
P_{\text{crit}} = \frac{V_S^2}{2X(1 - XB)}
\]  \hspace{1cm} (10.26)

As before, at maximum power \( \delta = 45^\circ \).

Equation (10.26) shows the increase in maximum power that can be attained by the application of shunt capacitors. There is also a proportionate increase in the receiving-end voltage at which the maximum power occurs, as shown by equation (10.25). However, the angle at which the maximum power occurs is still \( 45^\circ \).

**Example:** Consider the system shown in Figure 10.10, where 1000 MW is being delivered at the receiving end at 1.0 pu voltage. The sending-end voltage is also unity.
Fig. 10.10  Schematic of a system delivering a power of 10.0 pu on 100 MVA base at 1.0 pu sending-end voltage.

With no reactive support at the receiving end, the maximum power that can be transmitted is $V_S^2/(2X) = 6.25$ pu. However, since the theoretical limit of transmitted power at 1.0 pu sending- and receiving-end voltage is $1/0.08 = 12.5$ pu, it is possible to transmit 10.0 pu power with appropriate reactive support at the receiving end.

At 10.0 pu transmitted power at 1.0 pu sending- and receiving-end voltages, $\delta = \sin^{-1}(10 \times 0.08) = 53.13^\circ$. From the circle diagram, the required reactive support at the receiving end is obtained as

$$Q = 12.5 - 12.5 \cos 53.13^\circ = 5.0 \text{ pu}.$$  

$$\therefore B = \frac{Q}{V_R^2} = 5.0 \text{ pu}.$$  

Note that $\delta$ is greater than 45°. As noted earlier, in this operating state, with an incremental change in the connected load, a runaway situation would occur if the load is self restoring (constant power) type. It will be shown later that stable operation in this operating state is possible if the reactive support is by continuous control instead of in the form of fixed capacitors.

The operating point represents the solution shown in equation (10.24) corresponding to the negative sign before the radical. At $V_S = 1.0$, $B = 5.0$, $V_R$ is obtained from equation (10.24) as $V_R = 1.333$ or 1.0.

$dV_R/dQ$ is obtained from equation (10.19), at $Q = 0$ and after adjusting for $V_S$ and $X$, as –0.3 and 0.17, corresponding to the two solutions for $V_R$, respectively.

Note that at the solution point where $dV_R/dQ$ is negative, $V_R = 1.333$ and

$$\delta = \sin^{-1}(10.0 \times 0.08/1.333) = 36.88^\circ$$

Also, at $B = 5.0$, $P_{\text{max}} = 10.42$, from equation (10.26), and $V_R$ at maximum power = 1.167, from equation (10.25).

From the discussion presented above, it is clear that although some reactive support at the receiving end is necessary in order to enhance transmission capability and maintain a given voltage profile, with excessive reactive support, it is possible to closely approach the critical point. As has been demonstrated, the angle difference $\delta$ (in addition to $dV_R/dQ$) can be used as a good indicator. Although the critical angle is 45°, above 30° the operating point rapidly approaches the critical point with an incremental change in transmitted power, characterized by a rapid increase in $dV_R/dQ$. Therefore, a maximum operating angle of 30° is recommended.

**Steady-state Analysis of Voltage-Reactive Power Problem: Extension to Large Networks**

The power flow equations for a general network can be written in the following form:

$$f_P(x,y) = 0, \quad f_Q(x,y) = 0$$
where

\( x \) is the state vector: \( \theta, V \)

\( y \) is the control vector: \( P, Q, \) transformer taps, etc.

For a given set of system parameters and transformer tap settings, the above equations can be written as

\[
\begin{align*}
    f_p(\theta, V_G, V_L) - P &= 0 \\
    f_{QG}(\theta, V_G, V_L) - Q_G &= 0 \\
    f_{QL}(\theta, V_G, V_L) - Q_L &= 0
\end{align*}
\]  

where the subscripts \( G \) and \( L \) refer to the quantities pertaining to generator and load buses.

Linearizing (10.27), we have

\[
\begin{bmatrix}
    \Delta P \\
    \Delta Q_G \\
    \Delta Q_L
\end{bmatrix} =
\begin{bmatrix}
    \frac{\partial f_p}{\partial \theta} & \frac{\partial f_p}{\partial V_G} & \frac{\partial f_p}{\partial V_L} \\
    \frac{\partial f_{QG}}{\partial \theta} & \frac{\partial f_{QG}}{\partial V_G} & \frac{\partial f_{QG}}{\partial V_L} \\
    \frac{\partial f_{QL}}{\partial \theta} & \frac{\partial f_{QL}}{\partial V_G} & \frac{\partial f_{QL}}{\partial V_L}
\end{bmatrix}
\begin{bmatrix}
    \Delta \theta \\
    \Delta V_G \\
    \Delta V_L
\end{bmatrix}
\]  

which can be expressed as:

\[
\begin{bmatrix}
    \Delta P \\
    \Delta Q_G \\
    \Delta Q_L
\end{bmatrix} =
\begin{bmatrix}
    J_{P\theta} & J_{PVG} & J_{PVL} \\
    J_{QG\theta} & J_{QGVG} & J_{QGLV} \\
    J_{QL\theta} & J_{QLVG} & J_{QLVL}
\end{bmatrix}
\begin{bmatrix}
    \Delta \theta \\
    \Delta V_G \\
    \Delta V_L
\end{bmatrix}
\]  

(10.28)

Note that if the voltages at all the buses were held constant (\( \Delta V \equiv 0 \)), one would have to be concerned only with angle (synchronous) stability. At the angle stability limit \( J_{P\theta} \) is singular. When voltages are not held constant by active control, the network loading limit would probably be reached long before the angle stability limit is reached. Network loading limit is characterized by the system Jacobian becoming singular while \( J_{P\theta} \) remains nonsingular. From equation (10.29), with \( \Delta P = 0 \), and assuming that the generator bus voltages are maintained constant by excitation control, i.e., \( \Delta V_G = 0 \), we have

\[
\Delta V_L = \left[ J_{QLVL} - J_{QL\theta} J_{P\theta}^{-1} J_{PVL} \right] \Delta V_L
\]

or

\[
\Delta V_L = \left[ J_{QLVL} - J_{QL\theta} J_{P\theta}^{-1} J_{PVL} \right]^{-1} \Delta Q_L = S_{VLQL} \Delta Q_L
\]  

(10.30)

Similarly, with \( \Delta P = 0 \) and \( \Delta Q_L = 0 \), we have

\[
\Delta V_L = - \left[ J_{QLVL} - J_{QL\theta} J_{P\theta}^{-1} J_{PVL} \right]^{-1} \left[ J_{QLVG} - J_{QL\theta} J_{P\theta}^{-1} J_{PVG} \right] \Delta V_G = S_{VQLG} \Delta V_G
\]  

(10.31)

Equations (10.30) and (10.31) correspond to equations (10.19) and (10.21), respectively, for the two bus system. For operation within the network loading limit the elements of the matrices \( S_{VLQL} \) and \( S_{VQLG} \) must be positive. Operation beyond the maximum loading point is indicated by
one or more elements becoming negative. At the maximum loading point the matrices become singular.

The above two criteria are theoretically equivalent. The criterion expressed by equation (10.30) - - the \( \frac{dV_L}{dQ_L} \) criterion -- is, however, easier to implement. This is due to the fact that the elements of \( S_{VLQL} \) are obtained from the Jacobian elements that are computed and, therefore, readily available during the power flow solution by Newton's method. Computation of \( S_{VLVG} \) requires, in addition to the power flow Jacobian elements, the matrices \( J_{QLVG} \) and \( J_{PVG} \), which are not required and, therefore, not computed during a power flow solution, since the voltages at the generator buses are assumed constant unless the generator reactive limit is violated. Also, as can be seen from equation (10.31), the overall computation of \( S_{VLVG} \) is much more involved.

Note that \( S_{VLQL} \) is the lower diagonal block of the inverted power flow Jacobian. The Jacobian is extremely sparse. During the power flow solution, it is factored into lower and upper triangular matrices by employing one of the sparsity oriented factorization techniques. From these sparse triangular matrices, the elements of \( S_{VLQL} \) can be readily calculated. For practical purposes, only a small number of the elements of \( S_{VLQL} \) would be needed and these can be obtained with very little additional computational effort, since the bulk of the computation would already have been performed during the power flow solution.

Note that in the standard power flow formulation, the state vector is expressed as \([\Delta \theta, \Delta V, V]\), by suitably modifying the Jacobian elements, in order to reduce computational burden. The sensitivity elements obtained from the power flow Jacobian should therefore be multiplied by the corresponding voltages.

The magnitudes of the elements of \( S_{VLQL} \) can serve as a measure of the proximity to the maximum loading point. For normal secure operation these must be positive. (Note that in the analysis of the simple two bus system, due to the sign convention adopted for real and reactive power injection, a negative value of \( \frac{dV}{dQ} \) was indicative of operation in the normal range.) One or more negative values of the elements of \( S_{VLQL} \) would indicate operation in the undesirable region, corresponding to operating in the lower portion of the voltage-power characteristic in the simple two bus system.

For practical purposes, it is only necessary to look at the diagonal elements of \( S_{VLQL} \). In the operating region well removed from the loading limit, the elements will change very little with changes in the operating point. As a loading limit is approached, some of the elements would start increasing rapidly indicating impending voltage problem.

Approximate values of the elements of \( S_{VLQL} \) for operation well within the loading limit can be estimated by noting that under normal operating conditions, the bus voltage magnitudes are nearly equal and close to unity, and the angles between adjacent buses are small. Using these approximations, the following well-known relationship can be derived from equation (10.28).

\[
\Delta Q = B \Delta V
\]

or

\[
\Delta V = X \Delta Q
\]

where \( B \) is the imaginary part of the network admittance matrix with the rows and columns corresponding to the voltage controlled buses deleted, and \( X = B^{-1} \).
The elements of $X$ can be used as a guide in estimating the closeness of a given operating point to a critical point. If one or more of the elements of $S_{VLQL}$ deviates significantly from the corresponding elements of $X$, an impending voltage problem would be indicated. For practical purposes, only a small number of diagonal elements need be considered.

If the elements of $X$ are not readily available, the elements of $S_{VLQL}$ determined at a sufficiently low load level can serve as reference. As a practical guide, the elements of $S_{VLQL}$ at a given load level should not exceed twice their values determined at the low load level.

In a practical system, some of the load bus voltages may be held constant by voltage control equipment such as static var compensators. When the controls are operating within the reactive output limits, these buses should be treated as generator buses ($P$, $V$ bus) in forming the power flow Jacobian. When a generator reaches reactive limit, it is customary in a power flow solution to convert it to a load bus ($P$, $Q$ bus). As will be seen later, this can introduce considerable error in identifying impending voltage problem. At reactive limit, depending on the generator loading condition, the excitation may still remain on automatic control after having been adjusted manually in order to enforce the reactive limit, or the excitation might have reached the limit and therefore would be held constant. The proper procedure to handle a generator bus at reactive limit would therefore be as follows: If the generator is still on automatic excitation control it should be treated as a $P$, $V$ bus. If, on the other hand, the excitation has reached the limit, the power flow model should be augmented with the generator internal bus behind the synchronous impedance as a $P$, $Q$ bus while converting the terminal bus to a $P$, $Q$ bus. This would, of course, introduce some computational complexity.

Several examples will be given in order to illustrate the application of the criteria developed. First, consider the two-bus system studied earlier, which is repeated in Figure 10.11 for convenience. In Figure 10.11 the buses are numbered 1 and 2, with bus 1 taken as reference. The shunt admittance at bus 2 includes the equivalent admittance representing line charging. The added shunt admittance may therefore be positive or negative depending on the desired voltage level and line loading.

\[ P_1 + jQ_1 \rightarrow v_1 \] \[ v_2 \rightarrow P_2 + jQ_2 \]

Fig. 10.11 Schematic of a two-bus system.

The real and reactive power injected at the buses are:

\[
P_2 = -P_1 = \frac{V_1V_2}{X} \sin \theta
\]

\[
Q_1 = -\frac{V_1V_2}{X} \cos \theta + \frac{V_1^2}{X}
\]

\[
Q_2 = -\frac{V_1V_2}{X} \cos \theta + \frac{V_2^2(1-XB)}{X}
\]  

(10.34)

The required Jacobian elements are:
\[ J_{p\theta} = \frac{V_1 V_2}{X} \cos \theta, \quad J_{p\theta G} = \frac{V_2}{X} \sin \theta \]
\[ J_{p\theta L} = \frac{V_1}{X} \sin \theta, \quad J_{q\theta L} = \frac{V_1 V_2}{X} \sin \theta \]
\[ J_{q\theta G} = -\frac{V_2}{X} \cos \theta, \quad J_{q\theta L} = -\frac{V_1}{X} \cos \theta + \frac{2V_2(1-XB)}{X} \]

The sensitivity elements are calculated as:
\[ S_{VLQL} = -\frac{X \cos \theta}{2V_2(1-XB) \cos \theta - V_1} \]
\[ S_{VLVG} = \frac{V_2}{2V_2(1-XB) \cos \theta - V_1} \]

For \( X = 0.2, Q = 0 \), and assuming that the voltages are maintained at unity through proper adjustment of reactive support, the sensitivity elements are calculated at a number of power transfer levels and are plotted in Figure 10.12. Note that for the above specified conditions, the sensitivity elements reduce to:
\[ S_{VLQL} = \frac{X \cos \theta}{\cos 2\theta}, \text{ and } S_{VLVG} = \frac{1}{\cos 2\theta} \]

Fig. 10.12 Plots of \( S_{VLQL} \) and \( S_{VLVG} \) against \( P \).

From Figure 10.12 it can be seen that the sensitivity elements change very little until the power transfer increases to 2.0 pu. They increase to approximately twice their value at \( P = 0 \) when the power transfer is about 2.7 and \( \theta = -33^\circ \). If the reactive support were in the form of shunt capacitors, the maximum power point would be reached when \( P \approx 3.5 \) at \( \theta = -45^\circ \).
Since the system Jacobian becomes singular at a critical point, one or more eigenvalues of the Jacobian will be zero at a critical point. In the normal operating region, the eigenvalues will be positive. Therefore, the proximity to a critical point can be established by tracking the eigenvalues of the system Jacobian. However, in a complex power system it may be necessary to track a large number of eigenvalues. The minimum eigenvalue at a particular operating point may not be a reliable guide as to the proximity to loading limit. Depending on the change in the generation and loading pattern, a different eigenvalue may assume the minimum value at the next higher loading level. Tracking a large number of eigenvalues in a large system is impractical. In studies of small power systems eigenvalue tracking would be an alternative to the sensitivity approach in predicting proximity to the loading limit.

In the case of the simple system shown in Figure 10.11, the power flow Jacobian with constant $V_1$ is:

$$J = \begin{bmatrix}
V_1V_2\cos\theta & V_1\sin\theta \\
X & X \\
\frac{V_1V_2}{X}\sin\theta & \frac{2V_2(1-XB)}{X} - \frac{V_1}{X}\cos\theta
\end{bmatrix}$$

The eigenvalues of $J$ for $V_1 = V_2 = 1.0$ and $Q = 0$ are:

$$\lambda_{1,2} = \frac{\cos\theta \pm \sin\theta}{X}$$

The lower of the two eigenvalues, $\lambda_2$, is plotted against $P$ in Figure 10.13. Also shown in Figure 10.13 is the ratio of the eigenvalue of $J_{PQ}$ to $\lambda_2$.

![Fig. 10.13 Plots of $\lambda_2$ and $\lambda_{J_{PQ}}/\lambda_2$ against $P$.](image)

As a second example, consider the system shown in Figure 10.11 with mid-point reactive support by shunt elements (reactors or capacitors) in addition to the reactive support at the end. The arrangement is shown in Figure 10.14.
The real and reactive power injected at buses 1 and 2 are:

\[ P_1 = \frac{V_1 V_0}{X_1} \sin \theta_1 + \frac{V_1 V_2}{X_2} \sin \theta_{12} \]

\[ P_2 = \frac{V_1 V_2}{X_2} \sin \theta_{21} \]

\[ Q_1 = -\frac{V_1 V_0}{X_1} \cos \theta_1 - \frac{V_1 V_2}{X_2} \cos \theta_{12} + V_1^2 \left( \frac{1}{X_1} + \frac{1}{X_2} - B_1 \right) \]

\[ Q_2 = -\frac{V_1 V_2}{X_2} \cos \theta_{21} + V_2^2 \left( \frac{1}{X_2} - B_2 \right) \]

With \( V_0 \) as reference and given: \( V_0 = V_1 = V_2 = 1.0, \ X_1 = X_2 = X / 2, \ P_1 = Q_1 = Q_2 = 0, \) we have \( \theta_1 = \theta_{21} = \theta / 2, \) and \( B_1 = 2B_2. \)

From the system Jacobian, the \( V_1-Q_L \) sensitivity matrix is calculated from equation (10.30) as:

\[
S_{VLQL} = \begin{bmatrix}
-\frac{4}{X \cos(\theta/2)} + \frac{8}{X} - 2B_1 & -\frac{2}{X \cos(\theta/2)} \\
-\frac{2}{X \cos(\theta/2)} & -\frac{2}{X \cos(\theta/2)} + \frac{4}{X} - 2B_2
\end{bmatrix}^{-1}
\]

\( S_{VLQL} \) at various power transfer levels for \( X = 0.2 \) is shown in Table 10.1. It can be seen that the sensitivity elements change very little until the power transfer reaches about 2.0. Thereafter, they start increasing, first at a moderate rate and then very rapidly as \( P \) exceeds 3.0. The critical point is reached at about 3.75 when \( \theta = 45^\circ. \)

It is seen that the maximum power transfer with mid-point reactive support provided by shunt capacitors is not much higher than 3.5 obtained without mid-point support.
VOLTAGE STABILITY

Table 10.1 $S_{VQL}$ at various power transfer levels.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\theta$</th>
<th>$B_1$</th>
<th>$S_{VQL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>$\begin{bmatrix} 0.1 &amp; 0.1 \ 0.1 &amp; 0.2 \end{bmatrix}$</td>
</tr>
<tr>
<td>1.0</td>
<td>11.5</td>
<td>0.1</td>
<td>$\begin{bmatrix} 0.11 &amp; 0.11 \ 0.11 &amp; 0.21 \end{bmatrix}$</td>
</tr>
<tr>
<td>2.0</td>
<td>23.1</td>
<td>0.4</td>
<td>$\begin{bmatrix} 0.13 &amp; 0.14 \ 0.14 &amp; 0.26 \end{bmatrix}$</td>
</tr>
<tr>
<td>3.0</td>
<td>34.9</td>
<td>0.92</td>
<td>$\begin{bmatrix} 0.23 &amp; 0.28 \ 0.28 &amp; 0.45 \end{bmatrix}$</td>
</tr>
<tr>
<td>3.75</td>
<td>44.0</td>
<td>1.46</td>
<td>$\begin{bmatrix} 2.03 &amp; 2.83 \ 2.83 &amp; 4.06 \end{bmatrix}$</td>
</tr>
<tr>
<td>4.0</td>
<td>47.2</td>
<td>1.67</td>
<td>$\begin{bmatrix} -0.83 &amp; -1.22 \ -1.22 &amp; -1.66 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Problem: In the power transfer from System 1 to System 2 over a transmission line a portion of the power is tapped at an intermediate point as indicated in the one-line diagram below. Systems 1 and 2 are sufficiently large, so that the voltages $V_1$ and $V_2$ remain constant. The voltage at the intermediate point, $V_3$, is maintained at a specified level by appropriate reactive support by switched shunt capacitors as indicated.

Find the power limit, $P$, and the angle difference between Systems 1 and 2 ($\theta_1 - \theta_2$) for the following two cases:

(i) $P_2 = P_3 = P, \quad Q_2 = Q_3 = 0$
(ii) $P_2 = Q_2 = Q_3 = 0, \quad P_3 = P$

Assume $V_1 = V_2 = V_3 = 1.0, \quad X_1 = X_2 = 0.2$

(Hints: Take $V_1$ as reference ($\theta_1 = 0$); write the system Jacobian and apply equation 10.30)

Answers: (i) $P = 2.8$ (approx.), $\theta_1 = 81^\circ$ (ii) $P = 4.33$ (approx.), $\theta_1 = 60^\circ$.

In case (ii), what would be the limit if $X_2 = 100.0$?

Answer: $P = 3.53$ (approx.), $\theta_1 = 45^\circ$. 

10-20
How would the limits be affected if the load at the intermediate point were of constant impedance (resistance) type?

**Clarification of Certain Issues in Voltage Stability**

From the preceding discussions it should be clear that, in the case of a two bus system, for secure operation the operating point should be on the upper portion of the $PV$ curve (Fig. 4) and well away from the maximum power point. In this region the derivatives $dV_R/dP$ and $dV_R/dP$ are negative, i.e., the receiving-end (or the load bus) voltage decreases as the load is increased, as determined from the solution of the network equations and depicted in Figure 4. It should be emphasized that although meeting the above criterion is desirable for secure operation, it may not have anything to do with voltage stability.

Consider, for example, operation in the lower portion of the $PV$ curve (also known as the nose curve) shown in Figure 4. The plots would suggest that, at a given power factor, as load is dropped the voltage would go down farther, or conversely, as load is added the voltage would increase. If similar curves are plotted at various levels of reactive support at the load bus, they would suggest that, while operating in the lower portion, addition of reactive support would cause the voltage to drop. This of course cannot happen in a real system. This confusion disappears when one considers that the $PV$ curves of Figure 4 (and the corresponding $QV$ curves for given $P$) actually represent solutions of the network equations in the equilibrium state (the steady state solution), and as such they do not tell anything about what would happen if load ($P$ and/or $Q$) is added or dropped at a given operating point. The curves show two solutions (one high voltage and one low voltage) for each load level until the maximum power point is reached (where the two solutions coalesce) beyond which there is no solution. This means that as the load is increased the voltage at the high voltage solution will decrease and the voltage at the low voltage solution will increase. Also note that, although there is no solution beyond the critical point, in actual operation, depending on the characteristic of the load, load can still be added without causing voltage instability and collapse, although the demanded load will not be satisfied.

The misinterpretation of the $PV$ curve (or the nose curve) has been responsible for many of the misconceptions that still persist today. Even some of the popular definitions of voltage instability are based on this curve. As an example, consider the following definition found widely in the literature:

A power system is voltage unstable if an increase in reactive power input at any bus of the system causes a decrease in voltage at that bus.

In reality, voltage can never go down as reactive support is added, except under specific voltage control, and when that happens the operation is stable as will be shown later.

Analogous to the two bus system where the derivative $dV_R/dP$ (or $dV_R/dP$) becomes infinite at the maximum power point, in large networks the power flow Jacobian becomes singular at the maximum loading point (the loadability limit) -- the elements of the sensitivity matrix $S_{V_LQL}$ (equation 10.30) become indeterminate. Beyond the maximum power point some of the elements of the matrix change sign. The singularity of the power flow Jacobian has been liberally used in the literature as an indicator of voltage stability limit, although it has nothing to do with voltage stability except when a large number of conditions (often unrealistic) are satisfied simultaneously. Since the singularity of the Jacobian also signifies saddle node (or static)
bifurcation, the latter term has become synonymous with voltage stability limit. Voltage stability indices and voltage stability criteria have been defined based on these flawed concepts. At the maximum power point one of the eigenvalues of the power flow Jacobian or the reduced Jacobian becomes zero and, therefore, has served as an indicator of voltage stability limit in the voltage stability analysis software employing the so-called “modal analysis.”

Computer programs have been developed to accurately determine the maximum power point, also known as point of collapse, using various exotic algorithms, ignoring the fact that the maximum power point in a real system is very much dependent on the distribution of load (for a given generation pattern) the precise knowledge of which is not available and in which the system operator has no control. Given the uncertainty of the load distribution in the system it does not make sense to attempt to locate the “exact” “point of collapse.” Any efficient power flow program can locate the approximate loading limit under a given set of assumptions very quickly without resorting to an exotic algorithm.

A knowledge of the network loading limit is essential. It is also important to operate well within the limit so as to allow for contingencies and unexpected load increase. Assuring that the system voltages are maintained during normal, alert and emergency states is also important. These have been well known to system planners and operators, and extensive studies to ensure satisfactory performance have been performed as long as power systems have existed. Most of the commercially available voltage stability programs, being based on flawed thinking, do not therefore provide any additional benefits, although they may engender false security under certain circumstances when a voltage stability problem really exists.

**Voltage Stability Limit Determined by Singularity of the Power Flow Jacobian -- A Reality Check**

Singularity of the power flow Jacobian signifies steady-state limit, beyond which a solution would not exist. Over the years various attempts have been made to equate the singularity of the steady-state power flow Jacobian, or some other system Jacobian, with voltage stability limit. The fact that the singularity of the Jacobian can be a valid indicator of voltage stability limit has not been rigorously demonstrated, although many claims have been made to that effect. One important factor was missing in most of these works. The load dynamics, the driving force behind voltage instability, was not included in the analyses. Generally, a large scale system model was the starting point, which required writing a large number of equations using complex notations. In so doing it was easy to overlook the obvious.

We now present a collective review of these works. We will rely on assumptions made in the literature. We will consider the set of assumptions that allows the simplest formulation. Alternative assumptions will only increase the algebra. The end conclusion will not change. Briefly, the main assumptions are:

1. The generator internal impedances are rendered negligible due to the action of fast excitation controls that maintain the terminal voltages constant. As a consequence, the generator terminal voltage phasor can be assumed to coincide with the rotor position.

2. The generator mechanical input power is constant.

3. All loads are constant MVA.
These assumptions are rather drastic. We will not, however, try to justify the assumptions. Rather, our objective is to see where all these lead to, if the analysis results are interpreted correctly.

We will use a simple example system and follow the steps as generally followed in the literature. The system is shown in Figure 10.15. A single generator is connected to an infinite bus through a transmission line, and also serving an isolated load at the end of another transmission line. Although this is a very simple power network, it has all the essential features that can give rise to both voltage and angle instability. The reason for using this simple system is that the analysis steps and results will be easily tractable and, therefore, directly verifiable.

From Figure (10.15) we can write

\[
\frac{2H}{\omega_o} \dot{\theta}_g = P_m - \frac{V_o V_s}{X_1} \sin(\theta_g - \theta_t) - \frac{V_o V_L}{X_2} \sin(\theta_g - \theta_t) - D \dot{\theta}_g \tag{10.35}
\]

The load power balance equations are:

\[
P_L = \frac{V_G V_L}{X_2} \sin(\theta_g - \theta_t) \tag{10.36}
\]

\[
Q_L = \frac{V_G V_L}{X_2} \cos(\theta_g - \theta_t) - \frac{V_L^2}{X_2} + V_L^2 B
\]

Loads can be assumed as either constant MVA static load or some function of voltage. We will assume the former. The final conclusion will, however, be the same with either type of load.

We can linearize (10.35) and (10.36) around an operating point and, noting that \( \Delta P_L = \Delta Q_L = 0 \) for constant power static load, arrive at the following equation:

\[
\begin{bmatrix}
\Delta \dot{\theta}_g \\
\Delta \dot{\omega}_g \\
0
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
\omega_o A_1 & \omega_o A_2 \\
2H A_3 & 2H A_4
\end{bmatrix}
\begin{bmatrix}
\Delta \theta_g \\
\Delta \omega_g \\
\Delta \theta_t
\end{bmatrix} +
\begin{bmatrix}
0 \\
\Delta \dot{\theta}_g \\
\Delta \dot{\omega}_g
\end{bmatrix}
\tag{10.37}
\]

where \( A_1, A_2, A_3, \) and \( A_4 \) are functions of system parameters and initial state and given by

\[
A_1 = -\frac{V_G V_s}{X_1} \cos \theta_{gs} - \frac{V_G V_L}{X_2} \cos \theta_{gl},\quad A_2 = \left[ \begin{array}{c}
\frac{V_G V_L}{X_2} \cos \theta_{gl} \\
-\frac{V_G}{X_2} \sin \theta_{gl}
\end{array} \right]
\]
After eliminating the non-state variables, (10.37) reduces to
\[
\begin{bmatrix}
\Delta \dot{\theta}_g \\
\Delta \dot{\omega}_g \\
\end{bmatrix} = 
\begin{bmatrix}
0 \\
[\mathbf{A}_1 - \mathbf{A}_2 \mathbf{A}_4^{-1} \mathbf{A}_3]
\end{bmatrix}
\begin{bmatrix}
\omega_o \\
\frac{\omega_o}{2H}
\end{bmatrix}
\begin{bmatrix}
\Delta \theta_g \\
\Delta \omega_g
\end{bmatrix}
\]  
(10.38)

Equation (10.38) is of the form \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \). The stability of the system is determined by the eigenvalues of \( \mathbf{A} \). At stability limit one of the eigenvalues will become zero. This will be indicated by the determinant of \( \mathbf{A} \) becoming zero.

\[
\det \mathbf{A} = -\frac{\omega_o}{2H} \det \left[ \mathbf{A}_1 - \mathbf{A}_2 \mathbf{A}_4^{-1} \mathbf{A}_3 \right]
\]

The system power flow Jacobian is
\[
\mathbf{J} = 
\begin{bmatrix}
\mathbf{A}_1 & \mathbf{A}_2 \\
\mathbf{A}_3 & \mathbf{A}_4
\end{bmatrix}
\]

Using the expressions for \( \mathbf{A}_1, \mathbf{A}_2, \) etc. the determinant of \( \mathbf{J} \) can be computed as

\[
\det \mathbf{J} = \frac{V_g V_L}{X_1} \cos \theta_{gs} \det \mathbf{A}_4
\]  
(10.39)

The determinant of \( \mathbf{J} \) can also be expressed as

\[
\det \mathbf{J} = -|\mathbf{A}_4| |\mathbf{A}_1 - \mathbf{A}_2 \mathbf{A}_4^{-1} \mathbf{A}_3|, \quad \text{if } \mathbf{A}_4^{-1} \text{ exists}
\]

\[= \frac{2H}{\omega_o} |\mathbf{A}_4||\mathbf{A}|\]

or

\[|\mathbf{A}| = \frac{\omega_o}{2H} |\mathbf{A}_4| \]  
(10.40)

Equation (10.40) gives the impression that when \(|\mathbf{J}| \) is zero \(|\mathbf{A}| \) is zero, and therefore the singularity of \( \mathbf{J} \) signifies voltage stability limit. Actually, however, we can see from (10.39) that unless \( \theta_{gs} = 90^\circ \) (which is the angle stability limit), \(|\mathbf{J}| = 0 \) implies that \(|\mathbf{A}_4| = 0 \), and, therefore, singularity of the power flow Jacobian (or the saddle node or static bifurcation) means nothing as regards stability. Singularity of the power flow Jacobian signifies loadability limit and nothing else. As an extension, all related methodologies based on similar concepts, such as, modal analysis, point of collapse, static voltage stability indices, etc. [4 - 17] are therefore equally invalid.

The confusion clears up when one carries out the operations indicated in (10.38). After carrying out the indicated operations we obtain
VOLTAGE STABILITY

\[
A = \begin{bmatrix}
0 & 1 \\
-\frac{\omega_o \ V_o \ V_S}{2H} \cos \theta_{gs} & -\frac{\omega_o}{2H} D
\end{bmatrix}
\]

so that

\[
\det A = \frac{\omega_o \ V_o \ V_S}{2H} \cos \theta_{gs} \tag{10.41}
\]

The above expression clearly shows that, for the assumed load model, there can be no voltage instability, only angle instability, which occurs when \( \theta_{gs} \) reaches 90°. The same conclusion also follows from (10.39) and (10.40), from which we can write (10.41). This is a consequence of using a static load model in the analysis, and is to be expected. In fact, the mathematical steps just described are quite unnecessary when one observes that for constant power load, \( \Delta P_L = (\Delta P_2) \) is identically zero. Therefore (10.41) follows directly from (10.35).

Consider an example where \( V_G = 1.0, \ X_2 = 0.5, \ P_L = 1.6, \) and \( Q_L = 0. \) \( V_L \) is held at 1.0 by providing sufficient reactive support in the form of discrete capacitors. The network solution yields \( \theta_{gs} = 53°, \ B = 0.8. \) This represents operation in the lower portion of the PV curve, drawn with respect to constant \( V_G. \) It will be shown later that for real life constant power load, operation in this region is unstable. However, the above analysis shows otherwise.

If in the above analysis the effect of generator rotor flux dynamics were included, the results would have been even more dramatic. The movement of the eigenvalues with load increase would at first appear logical. However, instability will set in at a load level which would be typically well below the maximum as determined from a power flow model. If the computation is continued beyond the maximum power point, into the lower portion of the PV curve (corresponding to the low voltage solution), the eigenvalues will again show stable operation. (This is left as an exercise.) Obviously, this is not what one would experience with real life constant power load. This behavior can, however, be explained on the basis of the static constant power load model.

A constant power load is not a static load. This means it cannot jump instantaneously from one demand level to another as the demand changes. Similarly, following a system change, there is a momentary change in the load, which is then restored to the constant power level, either due to the nature of the load itself, or due to the action of some control mechanism. A definite time lag is involved in the process. It will be shown later that if the overall characteristic of the composite load is constant MVA, and the mechanisms that restore the load to constant MVA act much slower compared to the speed of system voltage control, voltage stability limit is the same as the power limit as obtained from a conventional power flow model.

**Voltage Stability: Effect of Load Characteristics**

The problem of reactive power and voltage control, i.e., maintaining an acceptable system voltage profile by providing adequate reactive supports at appropriate locations so as to meet system reactive needs efficiently, is well understood and reported extensively in the literature [1, 2]. It is not, however, well appreciated that maintaining a good voltage profile does not automatically guarantee voltage stability, and that voltage instability need not be associated with low voltages, although frequently it is.
Voltage instability is largely determined by load characteristics and the available means of voltage control. For true voltage instability at least a part of the total load must be of self-restoring (constant MVA) type. By the same token, voltage stability problem becomes less severe as static or voltage sensitive loads constitute a portion of the total load.

**System and load model**

To develop a basic understanding of voltage stability we will assume that voltage instability can be (although it need not be) studied in isolation, i.e., by separating it from the angle (synchronous) stability aspects. We will concentrate only on the dynamics that are likely to have appreciable impacts on voltage stability. The dynamics that affect voltage stability are the dynamics of the load and the dynamics of the voltage control devices, especially those of the generator excitation controls.

Important insights into the mechanisms of voltage instability and collapse can be obtained through the use of a simplified model, incorporating only the elements that are dominant in controlling these mechanisms. The conclusions drawn from the simplified analysis can then be verified by detailed simulations. The power system we will analyze is shown in Figure 10.16. A voltage source, which may be a generator terminal bus or an infinite system, is supplying a load over a transmission line. For simplicity, all resistances and line chargings have been neglected. However, any reactive support that might be present at the load end of the transmission line has been included.

![Phasor diagram](image)

**Fig. 10.16** Voltage source supplying load through transmission line. Phasor diagram drawn at unity pf

Load models that have been used in voltage stability studies can be generally classified into static and dynamic models. Although voltage stability is a dynamic phenomenon, a static model is perfectly acceptable if the load is static. Loads are considered static if they are functions of voltage, and can be modeled as such without needing any dynamic description (e.g., the use of differential equations). Examples are constant impedance and constant current loads. A common representation of static loads is

\[ P_L = P_0 \left( \frac{V_L}{V_{L,0}} \right)^a \]

\[ Q_L = Q_0 \left( \frac{V_L}{V_{L,0}} \right)^b \]  \hspace{1cm} (10.42)

By choosing appropriate exponent values, the above model can represent constant impedance or constant current loads. When both exponent values are zero, the load becomes a constant MVA load.
The most stringent load from the viewpoint of voltage stability is the load that maintains a constant MVA characteristic, either due to the nature of the load itself or due to the action of control mechanisms that are intended to maintain constant voltage at the load supply point, such as LTC’s, distribution voltage regulators, etc., thereby rendering any load constant MVA. (Note that even without voltage control action, certain apparently static loads, such as thermostatically controlled heating loads, due to their constant energy consumptions, tend to behave as constant MVA loads in the longer term.) We will therefore concentrate primarily on constant MVA loads, and show the effect on voltage stability when parts of the load possess characteristics other than constant MVA, such as constant impedance.

In a voltage stability analysis it is important to model the relevant dynamics of the load. As will be shown later, employing a static model for constant MVA loads can lead to erroneous and, often, misleading results. A constant MVA load is not a static load. This means it cannot jump instantaneously from one demand level to another as the demand changes. Following a change in the demand the load will at first change according to its instantaneous characteristic, such as constant impedance or current. It will then adjust the current drawn from the system until the load supplied by the system satisfies the demand at the final system voltage. Similarly, when there is a sudden change in the system voltage, such as following a disturbance, the load will change momentarily. It will then adjust the current (or impedance) and draw from the system whatever current is necessary in order to satisfy the demand. The process is not instantaneous. A definite time lag is involved. A load model, in order to be suitable for voltage stability analysis, must recognize this basic fact.

Since our purpose is to develop an understanding of the basic issues in voltage stability, we will adopt the simplest load model that satisfies the above characteristic of the constant MVA load so as to keep the analysis procedure simple while preserving the qualitative behavior of the system response. The simplified reduced order model of the induction motor -- the steady-state equivalent circuit along with the equation of motion, and assuming constant mechanical torque (see Chapter 7) -- can be used as a starting point. The model is used in the study of voltage stability in the presence of induction motor load, presented later in the chapter. Further simplification of the model by neglecting the stator resistance and all reactances yields the following form

\[ T_L \dot{G} = P_0 - V_L^2 G \quad (10.43) \]

The above applies to a unity power factor load. (For reactive power load a separate equation with \( G \) replaced by \( B \) may be used.) \( G \) plays the role of a load conductance which is adjusted to maintain constant power, and \( P_0 \) is the power set point. The model describes the basic dynamics, pertinent to voltage stability, of a wide variety of loads that recover, more or less exponentially, to constant power (e.g., static loads controlled by tap changers or other control devices), the specific time delay approximately reflecting the response characteristic of the particular load.

A general form of the above model can be found in the literature, and may be expressed as

\[ T_L \dot{G} = P(V) - V_L^2 G \left( \frac{V_L}{V_L^0} \right)^n \]

10-27
for the real power, and similarly for the reactive power. Note that the model is only valid for \( n \geq 1 \). For \( n < 1 \), it can be shown that the stability behavior of the model in certain operating ranges may be contrary to physical experience [4].

Other load models can be found in the literature on voltage stability. An example is a model using two components: a static component comprising a polynomial or exponential function of the load bus voltage, and a dynamic component comprising time derivatives of the load bus voltage and angle. The general form of this model is

\[
P_L = P(V) + k_1 \dot{\theta} + k_2 V_L \\
Q_L = Q(V) + k_3 \dot{\theta} + k_4 V_L
\] (10.44)

An assessment of the model in regard to its pertinence and validity in voltage stability studies will be provided later in the chapter.

**Region of attraction**

We now examine the region of attraction of the stable equilibrium point in the context of the system power voltage (PV) curves. Consider the simplified radial power system with constant sending-end voltage in Figure 10.16. Assuming, for simplicity, a unity power factor load, the PV curve of the post-disturbance system is shown in Figure 10.17. For a constant power load \( P_0 \), the load characteristic, as shown by the vertical line, would intersect the system PV curve at two points, A and B, corresponding to the two possible equilibrium points. Point A, on the upper portion of the system PV curve (the high voltage solution), is a stable equilibrium point, whereas point B, on the lower portion of the PV curve (the low voltage solution), is an unstable equilibrium point. This may be verified by linearizing equation (10.43) around the equilibrium points, and applying the condition for stability.

![Fig. 10.17 Power voltage curve of system shown in Fig. 10.16.](image)

It will be shown that if the state reached at the end of the disturbance lies anywhere on the portion \( V_{LoACB} \) of the PV curve, the post-disturbance system will be stable, i.e., the operating point will settle at the stable equilibrium point A. The region \( V_{LoACB} \), i.e., the region of the PV curve to the right of the unstable equilibrium point is the region of attraction of the stable equilibrium point of the post-disturbance system.

From the phasor diagram (Fig. 10.16)
VOLTAGE STABILITY

\[ V_L^2 = \frac{V_S^2}{(1 - BX)^2 + G^2X^2} \]  
\[ (10.45) \]

Equation (10.43) can therefore be expressed as

\[ T_L \dot{G} = \frac{\dot{P}_0}{(1 - BX)^2 + G^2X^2} \]

\[ = \frac{P_0X^2}{(1 - BX)^2 + G^2X^2} \]

\[ = \frac{P_0X^2(G - G_{10})(G - G_{20})}{(1 - BX)^2 + G^2X^2} \]  
\[ (10.46) \]

where

\[ G_{10,20} = \frac{V_S^2 \pm \sqrt{V_S^4 - 4P_0^2X^2(1 - BX)^2}}{2P_0X^2} \]

The two values of \( G \) correspond to the two steady-state solutions of \( G \) for the given power \( P_0 \). In Figure 10.17 these two solutions correspond to the equilibrium points B and A, respectively. The two voltage solutions, from (10.45) and noting that, in the steady state, \( \frac{V_L}{PGV} = L \), are

\[ V_L^2 = \frac{V_S^2 \pm \sqrt{V_S^4 - 4P_0^2X^2(1 - BX)^2}}{2(1 - BX)^2} \]

The higher value of \( G \) corresponds to the low voltage solution, and vice versa. This is verified by noting that

\[ V_L^2G_{20} = V_L^2G_{10} = P_0 \]

Equation (10.46) shows that if the initial state is to the left of the point A on the \( PV \) curve (\( G \) less than \( G_{20} \)), \( dG/dt \) is positive and therefore, the operating point will move to A, at which point \( dG/dt = 0 \). Similarly, if the initial state is anywhere on the portion ACB (\( G_{20} < G < G_{10} \)), \( dG/dt \) is negative and therefore the operating point moves to A. If the initial state is to the left of B, \( dG/dt \) is positive and the operating point moves further away from B. The region to the right of B is therefore the region of attraction of the stable equilibrium point A.

A physical explanation can be provided as follows: Consider an initial state in the lower portion of the \( PV \) curve but to the right of the unstable equilibrium point B. Since in this state the power delivered is greater than the set-point power \( P_0 \), the constant power control mechanism would decrease the current or admittance in order to bring the power down. This will, however, increase the power still further, since the voltage will increase at a faster rate with decrease in current or admittance in this region. The operating point will therefore move up the \( PV \) curve until point C is reached. From this point onwards the same control command will, however, decrease the power. The process will continue until the stable equilibrium point A is reached. Using the same argument it can be seen that starting from anywhere on the upper portion of the \( PV \) curve, the
operating point will move to the stable equilibrium point and that, starting from an initial state to the left of B, it will move further away. The region to the right of B is therefore the region of attraction of the stable equilibrium point A.

**Large disturbance voltage stability with constant source voltage**

It will now be shown that when the source voltage can be assumed constant, voltage stability is assured by the presence of a stable equilibrium point in the post-disturbance system. In a typical utility system, even when the loads possess constant MVA characteristics, the overall response time may be quite slow. Immediately following a disturbance all loads temporarily behave as static loads, e.g., constant impedance. It is either the nature of the load itself, or the action of some control mechanism that eventually restores the load to constant MVA. We will assume that the mechanisms by which much of the load would be rendered constant MVA act slowly compared to the speed of response of the voltage control equipment, which are primarily generator excitation controls. Voltages at the controlled buses, e.g., the generator terminal voltages would therefore be restored well before the overall load characteristics return to constant MVA. In situations when the bulk of the load has fast response characteristics, this assumption is not valid and can introduce considerable error. These cases will be discussed later.

Consider the system shown in Figure 10.16, initially supplying a load $P_1$. Assume a disturbance caused by opening one circuit of a double circuit transmission line. The pre- and post-disturbance system $PV$ curves are shown in Figure 10.18. For initial load power $P_1$, the operating point is A, the intersection of the vertical line at $P_1$ and the steady-state pre-disturbance $PV$ curve. Immediately following the disturbance the load will behave as constant impedance. If the system voltage can be assumed to be restored well before the load returns to constant power, the operating point will temporarily move to point $A'$, the intersection of the instantaneous load characteristics shown by the dashed curve a, and the steady-state post-disturbance $PV$ curve. Since $A'$ is within the region of attraction of the final steady-state equilibrium point B, the operating point would move to B and the system would be stable. The existence of a stable equilibrium point in the final post-disturbance system, therefore, guarantees voltage stability.

![Fig. 10.18 System PV curves and steady-state and instantaneous load characteristics.](image)

If, on the other hand, the initial load were $P_2$ (greater than the maximum power that can be supplied by the post-disturbance system), the operating point would temporarily move to $C'$ following the disturbance. Since there is no equilibrium point for the post-disturbance system at the assumed load, voltage collapse will ensue if the load attempts to maintain constant power. If
the speed of response of load recovery is slow, the collapse will not be immediate and sufficient
time may be available for applying corrective measures to restore stability.

Note that although we have used a two-bus system the results obtained apply to systems of any
size. This is because for every bus in a large system, PV curves similar to those shown in Figure
10.18 can be generated using a power flow program, and the arguments presented above can be
repeated.

The voltage stability limit for constant MVA load whose response time is slow compared to the
speed of response of the system voltage control, is the same as the power limit (in the literature
this has been called the steady-state voltage stability limit without proper qualifications),
obtained from the standard power flow model. In using the power flow model, the status of the
system voltage control is important and should be reflected in the model. If the generator
terminal voltages are not maintained by excitation control, i.e., when the generators are operating
at constant excitations, a modification to the power flow model would be necessary. It will be
shown later that under this condition the voltage stability limit and, therefore, the stable
equilibrium point are determined by constant voltage (generator field voltage) behind generator
synchronous reactance. Failure to recognize this may result in gross error.

Voltage stability limit when part of the load is static

Although voltage instability cannot occur for static load, the maximum power that can be
delivered at a given power factor is independent of the type of load. Beyond the point at which
maximum power occurs, the actual power delivered will decrease with increased demand. We
will now investigate the voltage stability limit when a portion of the load is static. Assuming, for
simplicity, unity power factor, and a combination of constant power and resistive load, the total
load can be expressed as

\[ P = V_L^2 (G + G_L) \]  

(10.47)

where \( G_L \) is the conductance of the resistive part of the load, and \( G \) is the conductance of the
constant power part of the load whose dynamics is given by (10.43)

From the power balance equations and using (10.47), we can write (also follows from the phasor
diagram of Figure 10.16)

\[ V_L (G + G_L)X = V_S \sin \theta \]  

(10.48)

\[ V_L (1 - BX) = V_S \cos \theta \]  

(10.49)

Linearizing (10.43), (10.48) and (10.49), and eliminating the non-state variables, we obtain

\[ T_L \dot{\Delta G} = -V_L^2 \left( 1 - \frac{2GX \tan \theta}{(G + G_L)X \tan \theta + (1 - BX)} \right) \Delta G \]

Using (10.48) and (10.49), the above reduces to

\[ T_L \Delta G = -V_L^2 \frac{G \cos 2\theta + G_L}{G + G_L} \Delta G \]  

(10.50)

For stability \( G \cos 2\theta + G_L > 0 \), which yields
where \( P_0 = V_L^2 G \) is the constant power part of the load.

Voltage stability limit is reached when

\[ \cos 2\theta = \frac{V_L^2 G_L}{P_0} \]  

(10.52)

Referring to Figure 10.19, assume that the load characteristic for a given load combination of constant power and resistive load is as shown by curve a. It intersects the system PV curve at points A and A'. It will now be shown that point A (although it is on the lower portion of the system PV curve) is a stable equilibrium point, whereas point A' is unstable.

In order to prove the above, we first increase the load until a limit is reached, i.e., until the load characteristic becomes tangent to the system PV curve. The load increase can be achieved by increasing either the constant power part or the resistive part of the load, or both.

![Diagram showing system PV curve and load characteristics for a combination of constant power and resistive load.](image)

Fig. 10.19 System PV curve and load characteristics for a combination of constant power and resistive load.

We have

\[ P_0 + V_L^2 G_L = \frac{V_L V_S}{X} \sin \theta \]  

(10.53)

Substituting \( V_L \) obtained from (10.49) into (10.53), we obtain

\[ P_0 + \frac{V_S^2 G_L}{(1 - BX)^2} \cos^2 \theta = \frac{V_S^2}{2X(1 - BX)} \sin 2\theta \]  

(10.54)

To obtain the limit, we take the derivative of either \( P_0 \) or \( G_L \) with respect to \( \theta \), and equate to zero. After some algebraic manipulations we arrive at the condition at limit as

\[ \cos 2\theta = \frac{-V_S^2 G_L}{P_0} \]  

(10.55)

which is the same as the voltage stability limit obtained above (10.52).
The limit shown on Figure 10.19 as point B, the point at which the load characteristic b is tangent to the system PV curve, is therefore also the voltage stability limit. It follows that the equilibrium point A is stable and A’ is unstable.

**Problem**

Referring to Figure 10.16, assume that the load is purely reactive and that the entire load is supplied locally by switched shunt capacitors so that there is no reactive flow on the transmission line, i.e., $V_R = V_S$. What would be the maximum load that can be supplied if the load is constant power. (Answer: $Q_{lim} = \frac{V_S^2}{2X}$)

**Extension to large networks**

The above result can be readily extended to large networks if we first relate the stability limit, as determined above, to the power flow Jacobian, modified to reflect the static part of the load. For the simple example considered above, the incremental power flow equations can be written as

$$
\begin{bmatrix}
\Delta P \\
\Delta Q
\end{bmatrix} =
\begin{bmatrix}
-\frac{V_S V_L}{X} \cos \theta & 2V_L G_L - \frac{V_S}{X} \sin \theta \\
\frac{V_S V_L}{X} \sin \theta & 2V_L (1 - BX) - \frac{V_S}{X} \cos \theta
\end{bmatrix}
\begin{bmatrix}
\Delta \theta \\
\Delta V
\end{bmatrix}
$$

Note that the conventional power flow Jacobian has been augmented by terms arising from the resistive part of the load.

We now determine the singularity condition of this modified Jacobian. The Jacobian is singular when the determinant is zero. After evaluating the determinant and carrying out the necessary algebraic manipulations, the condition for singularity is obtained as

$$
\cos 2\theta = -\frac{V_L^2 G_L}{P}
$$

which is the same as the condition for voltage stability limit. Therefore, the voltage stability limit, when the load contains static components, is indicated by the singularity of the modified Jacobian.

The extension to large networks follows directly from the above analysis. Power flow equations are first written, separating the voltage dependent part of the load from the constant power part, as shown in (10.56) and (10.57).

$$
P + P(V) = f(\theta, V) \quad (10.56)
$$

$$
Q + Q(V) = g(\theta, V) \quad (10.57)
$$

The linearized power flow equations are written as follows:

$$
\begin{bmatrix}
\Delta P \\
\Delta Q
\end{bmatrix} =
\begin{bmatrix}
J_{P\theta} & J'_{PV} \\
J_{Q\theta} & J'_{QV}
\end{bmatrix}
\begin{bmatrix}
\Delta \theta \\
\Delta V
\end{bmatrix}
$$

where
VOLTAGE STABILITY

\[ J'_{PV} = J_{PV} - \frac{\delta P(V)}{\delta V} \]

\[ J'_{QV} = J_{QV} - \frac{\delta Q(V)}{\delta V} \]

At the generator and other voltage controlled buses \( \Delta V' \)'s are zero, so that the Jacobian would be adjusted, as is customary in a power flow solution under perfect voltage control. It is important to account for any nonconformity of the voltage controlled buses. For example, when any of the generators reaches excitation limit, its terminal voltage is no longer constant. Instead, the constant voltage under fixed excitation is the voltage behind synchronous reactance (the field voltage), as noted earlier. This should be reflected in the Jacobian.

**Large disturbance voltage stability for mixed loads**

It has been shown that when a part of the total load is static, voltage stability limit can extend beyond the maximum power point on the system \( PV \) curve. For such loads stability can be maintained for initial pre-disturbance load greater than the maximum power capability of the post-disturbance system. This is because a stable equilibrium point may be possible in the post-disturbance system, although the actual power delivered will be reduced following the disturbance.

In order to illustrate this further, consider the case when about 50% of the load is resistive. The system \( PV \) curves and the steady-state load characteristics for this case are shown in Figure 10.20, in solid lines. For a total load \( P_1 \), A is the initial, pre-disturbance, operating point. The equilibrium points following the disturbance are shown on the post-disturbance \( PV \) curve at B and B'. (Recall that when a part of the load is static, both equilibrium points can be on the lower portion of the system \( PV \) curve.)

![Fig. 10.20 System \( PV \) curves and steady-state (solid lines) and instantaneous (dashed lines) load characteristics when part of the load is static.](image)

Using a similar argument as before, it can be concluded that the region to the right of B' (the unstable equilibrium point) is the region of attraction of the stable equilibrium point B.

Following a disturbance the operating point moves to \( A' \), the intersection of the post-disturbance \( PV \) curve and the instantaneous load characteristic, shown by the dashed line \( a' \). Since it is within the region of attraction of the stable equilibrium point B, the system will be stable. The approximate path of the operating point from initial to final state is marked by arrows as shown.
Note that if the initial total load is $P_2$, an equilibrium point would not exist in the post-disturbance system, and voltage instability would occur following the disturbance. Once again we can conclude that large disturbance voltage stability is assured by the existence of a stable equilibrium point in the post-disturbance system.

**Voltage stability for fast response load**

In the voltage stability analyses for constant MVA loads presented so far, it was assumed that the mechanisms that restore the loads to constant MVA act much slower compared to the mechanisms that restore system voltages, which are primarily the generator voltage controls. In many power systems this assumption may be unrealistic. When the load response speed is comparable to the speed of response of the voltage control devices, the assumption of constant system voltage can introduce considerable error.

A conceptual understanding of the issues involved in the voltage stability for such loads may be obtained through the use of “transient” system $PV$ curves. Figure 10.21 shows the steady-state and transient $PV$ curves when the load power is $P_0$. The transient curves, shown by dashed lines, were obtained using a fictitious voltage, held constant during the transient period, behind a fictitious reactance. In practice, it may not be a simple matter to estimate the correct value of this reactance. The present purpose is to provide a conceptual understanding of the issues involved and, as such, the transient curves are hypothetical. For the load $P_0$, the initial operating point is at $A$, the intersection of the pre-disturbance steady-state $PV$ curve, the transient $PV$ curve and the steady-state load characteristic, which is a vertical line at $P_0$ for the constant power load assumed.

![Fig. 10.21 Large disturbance voltage stability for fast response loads using “transient” $PV$ curves.](image)

Following the disturbance the operating point first moves to $A'$, the intersection of the post-disturbance transient $PV$ curve and the instantaneous load characteristic. Since $A'$ is within the region of attraction of the stable equilibrium point $B'$, on the transient $PV$ curve, the system will be transiently stable. The operating point will eventually settle at $B$, the intersection of the post-disturbance steady-state and final transient $PV$ curves and the steady-state load characteristic.

Figure 10.22 shows the case when the initial load is at a somewhat higher level, so that an intersection between the steady-state load characteristic and the post-disturbance system transient...
PV curve does not exist. [A steady-state equilibrium point, however, exists in the post-disturbance system, at the load level assumed.] It is evident that, without some sort of control action, voltage instability would occur following the disturbance. With appropriate control action, voltage stability can be maintained. For example, if sufficient capacitors are switched on promptly, stable operation can be restored as illustrated in the figure. In the illustration, the operating point first moves to \( A' \) following the disturbance. It would then move down the transient PV curve and, in the absence of any control action, voltage instability would follow. With prompt capacitor switching, the system state can be brought within the region of attraction of the stable equilibrium point on the new transient PV curve, so that it eventually moves to the stable equilibrium point C. Note that if the capacitor switching is delayed so that the operating point has moved beyond \( A'' \) on the transient PV curve, stable operation cannot be restored. This is because the initial state following the capacitor insertion would fall outside the region of attraction of the new stable equilibrium point. In some extreme situations it may be necessary to resort to SVC’s in order to maintain stability.

![Illustration of voltage instability following a large disturbance for a fast response load.](image)

Fig. 10.22 Illustration of voltage instability following a large disturbance for a fast response load.

The objective of the above discussion was to emphasize the need for a detailed analysis, considering both small and large disturbances, when the bulk of the load is composed of fast response loads and voltage stability may be in question. The analysis would require detailed modeling of generators and their control equipment, particularly, the excitation controls. No simple criterion can be employed in such situations. In non-critical situations, approximate estimates of voltage stability limits for such loads may be obtained based on steady-state analyses, using a constant generator internal voltage behind a conservatively estimated reactance, in place of the constant generator terminal voltage as used in the conventional power flow model.

**Voltage Stability Using Static Load Models**

In order to keep the algebra simple, the load will be assumed to be purely reactive. Also, for simplicity, only the generator field flux dynamics is considered, and constant field voltage (manual excitation control) is assumed. The generator equations, in simplified form and using standard notations (see Chapter 5), are

\[
T_{do}'e_q' = E_{fd} - (x_d - x'_d)i_d - e_q'
\]

(10.59)
VOLTAGE STABILITY

\[ e'_q = e_q + x'_d i_d \]  \hspace{1cm} (10.60)

\[ 0 = e_d - x_q i_q \]  \hspace{1cm} (10.61)

Referring to Figure (10.16), since the load is assumed to be purely reactive, from equation (10.42),

\[ Q_L = Q_0 \left( \frac{V_L}{V_{L0}} \right)^b = V_L I_L \]  \hspace{1cm} (10.62)

where

\[ I_L = I + V_L B \]

Also

\[ i_d = I, \quad i_q = 0, \quad e_d = 0 \quad \text{and} \quad e_q = V_s = V_L + IX \]

After linearizing equations (10.59) - (10.62) around the operating point, and eliminating the non-state variables, the following equation is obtained:

\[ T_{d0} \Delta e'_q = - \frac{V_L^2 (1 - BX_d) + X_d Q_0 (b-1)}{V_L^2 (1 - BX'_d) + X'_d Q_0 (b-1)} \Delta e'_q \]  \hspace{1cm} (10.63)

where

\[ X_d = X + x_d \quad \text{and} \quad X'_d = X + x'_d \]

For normal values of \( B, BX_d \) (or \( BX'_d \)) \(< 1\). Equation (10.63) shows that, for \( b = 1 \) or 2, operation is always stable. This means that for static loads, such as constant impedance or constant current loads, there cannot be voltage instability. For \( b = 0 \), equation (10.63) becomes

\[ T_{d0} \Delta e'_q = - \frac{V_L^2 (1 - BX_d) - X_d Q_0}{V_L^2 (1 - BX'_d) - X'_d Q_0} \Delta e'_q \]

which can be written as

\[ T_{d0} \Delta e'_q = - \frac{1 - (B + B_L) X_d}{1 - (B + B_L) X'_d} \Delta e'_q \]  \hspace{1cm} (10.64)

where \( B_L \) is the equivalent admittance of the constant reactive power load \( Q_0 = V_L^2 B_L \).

Equation (10.64) shows that operation is stable for \( (B + B_L) X_d < 1 \) as well as for \( (B + B_L) X'_d > 1 \), and unstable for \( (B + B_L) X_d > 1 > (B + B_L) X'_d \). This means that, as the loading is increased from zero, there will be a stable region, followed by an unstable region, then another stable region. This is illustrated on the (reactive) power/voltage curve in Figure 10.23, drawn for constant generator terminal voltage, held at unity by manual excitation control, and using the parameter values \( x_d = 1.5, x'_d = 0.3, X = 0.3, B = 0 \).
Fig. 10.23 Stable and unstable regions as obtained from constant (reactive) power static load model.

Note that, according to this model of the constant (reactive) power load, the entire lower portion of the power-voltage curve, drawn with respect to the generator terminal voltage is stable. This conclusion will not change even if excitation and other control actions are included in the analysis. For example, in Reference 20, voltage stability analysis using a detailed generator and control system model, but static model for the constant power load has been reported. The analysis identified the entire lower portion of the $PV$ curve as stable. A consideration of the physical behavior of a constant power load will show [21] that this is not possible. The cause of the anomaly, which is due to the use of a static model to represent a dynamic element that determines the stability behavior, should be evident from the above analysis. The analyses presented in subsequent sections will confirm that, if a realistic model were employed for the constant power load, the second stability region would not exist.

When the effect of the excitation control is included in the system model, the stability region will extend considerably. The point at which instability sets in will depend on the dynamic response speed of the constant power load. For loads with slow dynamics, the stability limit will occur at the same point as the maximum power determined from a power flow analysis (i.e., at the tip of the $PV$ curve).

The above analysis demonstrates a fundamental problem with the traditional static load model, described by a second order polynomial (the so-called ZIP model), widely used in large scale dynamic simulation of power systems. Although the constant power part would usually either be zero or constitute a small fraction of the total load, on those rare occasions when a significant part of the load is specified as constant power load, there is the potential for misleading results and/or computational problems.

A practical illustration of the problem of using a static model for constant power load in a dynamic simulation is a case study reported in Reference 22. In the second example of that paper, the load, modeled as a constant power static load, was ramped up and down while operating in the lower portion of the $PV$ curve. No instability was evident. Although the results are computationally correct, the apparently stable operation is a consequence of using a static model for the constant power load in a dynamic simulation, as explained in the analysis presented above.
VOLTAGE STABILITY

Constant Power Dynamic Load with Fixed Generator Field Voltage

Again, for simplicity, a purely reactive load is assumed. The generator equations are as given in

\[ T_L \dot{B}_L = Q_0 - V_L^2 B_L \]  

(10.65)

Linearizing equations (10.59) - (10.61), and (10.65), and eliminating the non-state variables, the

state-space formulation is obtained as

\[
\begin{bmatrix}
\Delta e'_q \\
\Delta B_L
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
\Delta e'_q \\
\Delta B_L
\end{bmatrix}
\]

where

\[
a_{11} = - \frac{1}{T_d'} \frac{1 + X_d' B_L'}{1 + X_d' B_L'} \quad a_{12} = - \frac{V_L}{T_d'} \frac{x_d - x_d'}{1 + X_d' B_L'}
\]

\[
a_{21} = - \frac{2V_L B_L'}{T_L (1 + X_d' B_L')} \quad a_{22} = - \frac{V_L^2}{T_L} \left( 1 - \frac{2X_d' B_L'}{1 + X_d' B_L'} \right)
\]

and

\[ B_L' = B_L - B \]

For stability,

\[-(a_{11} + a_{22}) > 0 \quad \text{and} \quad a_{11} a_{22} - a_{12} a_{21} > 0 \]

The first condition is automatically satisfied, provided the second condition is satisfied. After

carrying out the indicated operations and simplifying,

\[
a_{11} a_{22} - a_{12} a_{21} = \frac{V_L^2}{T_L T_d'} \left( 1 - X_d (B_L + B) \right) \left[ 1 + X_d' (B_L - B) \right]
\]

Practical values of \( B \) are such that \( BX_d' \ll 1 \). Therefore, the condition for stability is

\[(X + x_d)(B_L + B) < 1\]

This is the same condition as would be arrived at by assuming constant voltage behind the

synchronous reactance \( x_d \) (i.e., the field voltage). Therefore in the absence of active excitation

control, generator flux dynamics have no effect on voltage stability limit. The stability limit is
determined by the fixed voltage behind synchronous reactance.

The above analysis demonstrates the importance of operating well within the excitation limit when

the load behavior is constant MVA. In the steady-state analysis of voltage stability, it is common

practice to convert a generator bus from a \( P, V \) bus into a \( P, Q \) bus when the generator reactive limit

corresponding to a given \( P \) is reached. Although this forces the terminal voltage to decrease as the

system loading increases, the situation would be far worse if the computation were carried out with

fixed generator field voltage, which would be the correct procedure. An operating point which has

sufficient stability margin with respect to the controlled generator terminal voltage may be well

within the unstable region with respect to the fixed generator field voltage. When the excitation hits

limit, a stable operating point may suddenly be pushed well into the unstable region.
As an illustration, consider the system of Figure 10.16, supplying a unity power factor load $P_L = 1.0$, with parameter values $X = 0.3$, and $x_d \approx x_q \approx 1.5$. Assume $V_S = 1.0$ and $V_L$ is held constant at unity by providing sufficient reactive support by shunt capacitors at the load bus. For this operating condition, the field voltage, $E_{fd}$, is calculated as 1.94, and $B = 0.1535$. The $PV$ curves corresponding to operation at constant generator terminal voltage and constant field voltage calculated at the assumed load ($P_L = 1.0$) are shown in Figure 10.24. Curve a applies if the generator terminal voltage is under active excitation control whose response speed is much faster than the speed of load recovery, while curve b applies if the excitation is at limit and, therefore, unable to provide voltage control. It is evident that although the operating point is on the upper part of the $PV$ curve for constant terminal voltage, and well within the voltage stability limit under active excitation control, it can be pushed well into the unstable region if the field voltage reaches its upper limit. This is because, as the above analysis has shown, when the field voltage is held constant, voltage stability is determined by the fixed field voltage, not the generator terminal voltage.

![Fig. 10.24 Power-voltage characteristics](image)

(a) for constant generator terminal voltage, (b) for constant generator field voltage

**Induction Motor Load**

Induction motors require special consideration in voltage stability analysis. An induction motor with constant load torque will generally become unstable well before the voltage stability limit as determined for loads specified simply by the MVA demand and power factor. Also, the response speed of a motor load is fast and may be comparable to the speed of response of the voltage control equipment. However, in situations when the source voltage may be assumed to remain more or less
constant, induction motor loads can be correctly accounted for in voltage stability studies by steady-state analyses.

For the purpose of illustration, a simplified motor equivalent circuit will be used and shown in Figure 10.25a. For simplicity, the mechanical load torque, \( T_m \), will be assumed to be constant. The stability condition will first be derived from a dynamic analysis, considering only the basic dynamics of the motor. The results will then be used to validate the steady-state analysis. [Later it will be shown that the fundamental conclusion will not change when a more detailed dynamic model of the motor is employed.]

![Fig. 10.25 Constant voltage source supplying induction motor load](image)

(a) represented by simplified equivalent circuit, (b) Thevenin equivalent of a

The basic equation representing the motor dynamics is

\[
\frac{2H}{\omega_o} \dot{\omega} = T_e - T_m \tag{10.66}
\]

where \( H \) is the inertia constant. Since the slip, \( s \), is given by \( s = (\omega_o - \omega) / \omega_o \), where \( \omega_o \) and \( \omega \) are the synchronous and motor speeds, respectively, we have, \( \omega_o \dot{s} = -\dot{\omega} \). Also, \( T_e = E^2(s/r) \). Therefore equation (10.66) reduces to

\[
T_L \dot{G} = T_m - E^2 G \tag{10.67}
\]

where \( T_L = 2Hr \), and \( G = s/r \). Note that the structure of equation (10.67) is similar to that of (10.43).

The circuit of Figure 10.25a is converted into the Thevenin equivalent shown in Figure 10.25b, where

\[
\begin{align*}
V'_s' &= \frac{V_s}{1 - BX_1} \quad \text{and} \quad X'_1 = \frac{X_1}{1 - BX_1}
\end{align*}
\]

The power flow equations can be written as

\[
\begin{align*}
E^2 G &= \frac{V'_s'E}{X} \sin(\theta + \alpha) \\
0 &= \frac{V'_s'E}{X} \cos(\theta + \alpha) - \frac{E^2}{X}
\end{align*}
\]
VOLTAGE STABILITY

where

\[ X = \frac{X_1}{1 - BX_1^2} + X_2 \]

Linearizing and eliminating the non-state variables, the following is obtained:

\[ T_e \Delta \dot{G} = -E^2 \cos 2(\theta + \alpha) \Delta G \]  

(10.68)

Therefore, operation is stable for \( \theta + \alpha < 45^\circ \), and the stability limit is reached when \( \theta + \alpha = 45^\circ \)

**Modification of the power flow Jacobian to account for induction motor load**

It will now be shown that the same stability condition as implied by (10.68) will also be obtained from a steady-state analysis by appropriately modifying the conventional power flow Jacobian to reflect the power-voltage characteristic of the induction motor load. From the equivalent circuit, it is apparent that the reactive power requirement of an induction motor load, for a given load, increases as the supply voltage decreases, and vice versa.

Since the load torque is assumed constant, the real power drawn by the motor from the system is approximately constant in the steady state. From the equivalent circuit, the complex power drawn by the motor is given by

\[ P_L + jQ_L = \hat{V}_L \hat{I}_L^* = \hat{V}_L \left[ \frac{\hat{V}_L}{r/s + jX_2} \right]^* \]

which yields

\[ P_L = \frac{V_L^2 (r/s)}{(r/s)^2 + X_2^2} \]  

(10.69)

\[ Q_L = \frac{V_L^2 X_2}{(r/s)^2 + X_2^2} \]  

(10.70)

Linearizing (10.69) and (10.70), and eliminating \( \Delta s \), the following is obtained:

\[ \Delta Q_L = k \Delta V_L \]  

(10.71)

where

\[ k = -\frac{2V_L X_2}{(r/s)^2 - X_2^2} \]

Note that, since \( \tan \alpha = (s/r)X_2 \) (see phasor diagram of Fig. 10.25b), \( k \) can also be expressed as

\[ k = -\frac{2V_L \sin^2 \alpha}{X_2 \cos 2\alpha} \]

The power balance equations at the load bus for the power system are

\[ P_L = \frac{V_S V_L}{X_1} \sin \theta \]

\[ Q_L = \frac{V_S V_L}{X_1} \cos \theta - \frac{V_L^2 (1 - BX_1)}{X_1} \]

10-42
The incremental power flow equations can therefore be written as

\[
\begin{bmatrix}
\Delta P \\
\Delta Q
\end{bmatrix} =
\begin{bmatrix}
J_{p\theta} & J_{p\varphi} \\
J_{q\varphi} & J'_{q\varphi}
\end{bmatrix}
\begin{bmatrix}
\Delta \theta \\
\Delta V
\end{bmatrix}
\]

where

\[
J_{p\theta} = -\frac{V_S V_L}{X_1} \cos \theta \\
J_{p\varphi} = -\frac{V_S}{X_1} \sin \theta \\
J_{q\varphi} = \frac{V_S V_L}{X_1} \sin \theta \\
J'_{q\varphi} = \frac{2V_L (1 - BX_1)}{X_1} - \frac{V_S}{X_1} \cos \theta + k = J_{q\varphi} + k
\]

where \(J_{q\varphi}\) is the corresponding element of the conventional power flow Jacobian. Therefore, the effect of the induction motor load is to modify the power flow Jacobian. The modification is indicated by the addition of the extra term to \(J_{q\varphi}\).

The \(\Delta V - \Delta Q\) relationship is now derived by setting \(\Delta P = 0\) in (10.72), which yields

\[
\Delta Q = \left[J'_{q\varphi} - J_{q\varphi} J_{p\theta}^{-1} J_{p\varphi}\right] \Delta V_L
\]

After carrying out the indicated operations, and noting that \(V_L (1 - BX_1) \cos \alpha = V_S \cos (\theta + \alpha)\), from the phasor diagram of Figure 10.25b, and applying some algebraic manipulations, the following is obtained:

\[
\frac{\Delta V_L}{\Delta Q} = \frac{X_1 \cos \theta \cos 2\alpha}{V_S \cos 2(\theta + \alpha)}
\]

In a static analysis, stable operation would be indicated by the above expression being positive. It is positive when \(\theta + \alpha < 45^\circ\). At stability limit \(\Delta V_L/\Delta Q\) is infinity, which occurs when \(\theta + \alpha = 45^\circ\). As \(\theta + \alpha\) exceeds \(45^\circ\), \(\Delta V_L/\Delta Q\) becomes negative, indicating instability. So far, the results agree with those obtained from the dynamic analysis presented earlier. Note, however, that as \(\alpha\) exceeds \(45^\circ\), \(\Delta V_L/\Delta Q\) becomes positive again and therefore the static analysis would indicate stability, although the operation is clearly unstable. Actually, in the analysis presented, this condition is easy to detect.

In large power system studies using a computer, a more complex equivalent circuit than that used here would probably be employed, and the computation would probably be performed in terms of the motor slip, rather than the angle \(\alpha\). The possibility of converging on the wrong value of slip during initialization is there, especially under low voltage conditions. Consider the following example, where the parameter values are:

\[
X_1 = X_2 = 0.3 \quad r = 0.03 \quad V_S = 1.0 \quad V_L = 0.95 \quad \text{and} \quad P_L = 1.5
\]

Assume that the entire reactive requirement of the motor (\(Q_L\)) is supplied by a static capacitor of susceptance \(B_2\), so that, in the steady state, the load power factor is unity. Also assume that \(B_1\) supplies the reactive requirement to maintain \(V_L\) at the specified value for the unity power factor load.

Solving the system equations, \(\theta = 28.3^\circ\) and \(B_1 = 0.243\). From (10.69) and (10.70), the motor slip is calculated as \(s = 0.093\) or \(0.108\), corresponding to the two possible operating points. The corresponding values of \(B_2\) are 1.545 and 1.789.
\[ \Delta V_l / \Delta Q \] for the two operating points calculated from (10.73) are -0.024 and 0.022, respectively. This would suggest that the second operating point is stable, whereas the dynamic analysis will show both operating points to be unstable. However, it would be a simple matter to detect this possibility and guard against it.

**Effect of finite speed of voltage control**

In the analysis of voltage stability due to induction motor load presented so far, the assumption was made that the system voltage would be maintained constant, either by virtue of the system being large compared to the motor load or due to the action of the fast acting voltage control equipment. If the motor load constitutes a substantial portion of the total load, the assumption of constant source voltage is not valid. Also, when the actual response characteristic directly influences the stability performance, such as in a large disturbance situation, when the duration of the disturbance is appreciable, a detailed motor representation, including the internal flux dynamics, may be desirable, if the study results are not to be overly pessimistic. It will be shown later that, when the source voltage is constant, the level of modeling details does not influence small-disturbance stability results, provided the basic motor dynamics is included in the model.

The effect of the finite speed of response of the voltage control equipment on motor stability will be demonstrated using a simplified representation of the combined generator-excitation system as shown in Figure 10.26. This simple approach is followed here since the objective is to identify the voltage stability issues rather than to calculate the exact stability limit and/or detailed system response.

![Schematic of induction motor load and supply voltage controlled with finite time lag](image)

In the simplified representation, the system voltage is held constant by adjusting an internal voltage, \( E_g \), behind an equivalent reactance, \( x_g \), through a first order time delay. In order to keep the algebra simple, the shunt support at the load bus has been neglected.

The basic motor dynamics is given by equation (10.67). From Fig. 10.26, the voltage control equation can be written as

\[ T_e \dot{E}_g = V_{ref} - V_S \] (10.75)

Writing \( \delta_1 = \delta + \alpha \), and \( \theta_1 = \theta + \alpha \), the power balance equations can be written as

\[
E^2 G = \frac{E_g E}{X_g} \sin \delta_1 = \frac{V_S E}{X} \sin \theta_1
\]

\[
0 = \frac{E_g E}{X_g} \cos \delta_1 - \frac{E^2}{X} = \frac{V_S E}{X} \cos \theta_1 - \frac{E^2}{X}
\]

where
\[ X_g = x_g + X_1 + X_2 \quad \text{and} \quad X = X_1 + X_2. \]

Linearizing and eliminating the non-state variables,

\[
\begin{bmatrix}
\Delta E_g \\
\Delta G
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta E_g \\
\Delta G
\end{bmatrix}
\]

where

\[
a_{11} = -\frac{\cos \delta_1}{T_c \cos \theta_1} \quad a_{12} = -\frac{EX \sin \theta_1}{T_c} \left(1 - \frac{\sin^2 \delta_1}{\sin^2 \theta_1}\right)
\]

\[
a_{21} = -\frac{2EG^2X_g \cos^2 \delta_1}{T_L \sin \delta_1} \quad a_{22} = -\frac{E^2 \cos 2\delta_1}{T_L}
\]

For stability,

\[-(a_{11} + a_{22}) > 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} > 0\]

which yield

\[
\frac{\cos \delta_1}{T_c \cos \theta_1} + \frac{E^2}{T_L} \cos 2\delta_1 > 0 \quad \text{and} \quad \frac{E^2 \cos \delta_1}{T_c T_L \cos \theta_1} \cos 2\theta_1 > 0
\]

The necessary condition for stability is therefore \(\theta_1 < 45^\circ\). This is also sufficient when \(T_L \gg T_c\).

Note that, as \(T_L \to 0\), stability condition becomes \(\delta_1 < 45^\circ\). Since, in the above analysis, a fictitious internal voltage behind a fictitious reactance was assumed, the need for a detailed analysis in situations where \(T_L\) is comparable to \(T_c\) is obvious.

**Stability analysis using detailed induction motor model**

In this analysis the motor model will include only the rotor flux dynamics. Assuming the motor is being supplied from an infinite source of voltage \(V_S\), the two-axes motor transient equations, using \(V_S\) as reference and neglecting stator resistance, are (see Chapter 7)

\[
T_o'\dot{e}_D' = -e_D' - (x-x')\dot{i}_Q + T_o'\omega_s e_Q' \tag{10.76}
\]

\[
T_o'\dot{e}_Q' = -e_Q' + (x-x')\dot{i}_D - T_o'\omega_s e_D' \tag{10.77}
\]

\[
T_e' = e_D'i_D + e_Q'i_Q \tag{10.78}
\]

\[
e_D' = e_D + x'i_Q \tag{10.79}
\]

\[
e_Q' = -x'i_D \tag{10.80}
\]

where the terms are defined in Chapter 7.

In the above equations positive directions of current and torque correspond to motor action. Note that, since \(V_S\) is reference, \(e_D = V_S\) and \(e_Q = 0\).

The equation of motion is given by (10.66). For simplicity, the load torque is assumed constant. Stability of an operating point can be assessed by linearizing the above set of equations and applying the stability conditions (i.e., the eigenvalues must have negative real parts). The details of
the algebra are omitted here, and left as an exercise. It will be seen that, for the assumed constant load torque, the stability limit occurs when $T_e$ is maximum, at which point

$$s = \frac{x}{x'T_o'\omega_o}$$

(10.81)

$$T_{e\max} = \frac{V_s^2(x-x')}{2xx'}$$

(10.82)

$$e_D' = -e_Q' = \frac{V_s(x-x')}{2x}$$

(10.83)

(i.e. $E' = \sqrt{e_D'^2 + e_Q'^2}$ lags $V_S$ by $45^\circ$)

Substituting the expressions for $x$, $x'$ and $T_o'$ (see Chapter 7) in the above equations, it is seen that there is close agreement with the results obtained earlier using the basic dynamic model. The small discrepancy is due to the simplified equivalent circuit used in the basic model, where it was assumed that $x_m >> x_s$ or $x_r$. As in the case of the synchronous machine without excitation control, the electrical transients have no effect on small-disturbance stability of induction motors when the source voltage is constant. The performance difference between the basic dynamic model and a more detailed model will only be noticeable under large disturbance conditions, and when the duration of the disturbance is appreciable. The effect of the electrical transients is to slow down the motor slip excursions somewhat, thereby aiding stability and/or buying some time for possible corrective action. The effect is more pronounced in large machines than in small ones, due to design characteristics [33].

**Voltage Stability Using Type-Two Load Model**

The general form of the load model is given by equations (10.44), reproduced here for convenience

$$P_L = P(V) + k_\theta \dot{\theta} + k_\dot{\theta} \ddot{V}_L$$

$$Q_L = Q(V) + k_\theta \dot{\theta} + k_\dot{\theta} \ddot{V}_L$$

The static part represents the load characteristic in the steady state. A constant static part would represent a constant power load. However, the dynamic behavior of this model would be distinctly different from that of the static constant power load analyzed earlier. The complete right hand side constitutes the load model and it is a *bona fide* dynamic model. However, if the static part reflects a truly static load, such as constant impedance, the stability behavior of the model would be similar to that of the static load (i.e., there would be no voltage instability). This will be evident from the analysis that follows.

The power balance equations can be written as

$$P_L = \frac{V_s^2 V_L}{X} \sin \theta$$

(10.84)

$$Q_L = \frac{V_s^2 V_L}{X} \cos \theta - \frac{V_L^2(1-BX)}{X}$$

(10.85)
Note that, when both $k_1$ and $k_3$ are zero, then either $k_2$ or $k_4$ must also be zero to uniquely determine $V_L$. Similarly, when $k_2$ and $k_4$ are zero, either $k_1$ or $k_3$ must also be zero.

Assume $P(V) = P_0 + V_L^2 G_L$, and $Q(V) = 0$. In the steady state

$$P_0 + V_L^2 G_L = \frac{V_S V_L}{X} \sin \theta$$

$$V_S \cos \theta = V_L (1 - BX)$$

For simplicity, two special cases will be considered, first with $k_3 = 0$, and then with $k_4 = 0$.

Case (i) $k_3 = 0$

Linearising and eliminating the non-state variables, the system in state space form is

$$\begin{bmatrix} \Delta \dot{\theta} \\ \Delta \dot{V}_L \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta \theta \\ \Delta V_L \end{bmatrix}$$

where

$$a_{11} = \frac{V_S V_L}{k_1 X} \left[ \cos \theta + \frac{k_2}{k_4} \sin \theta \right]$$

$$a_{12} = \frac{V_L}{k_1 X} \left[ \frac{\sin \theta}{\cos \theta} + \frac{k_2}{k_4} - 2 G_L X' \right]$$

$$a_{21} = -\frac{V_S V_L}{k_4 X} \sin \theta$$

$$a_{22} = \frac{V_L}{k_4 X'}$$

and

$$X' = \frac{X}{1 - BX}$$

For stability,

$$-(a_{11} + a_{22}) > 0 \quad \text{and} \quad a_{11} a_{22} - a_{12} a_{21} > 0$$

Provided $k_1$ is negative and $k_4$ is positive, the first condition is satisfied for $\theta$ up to $90^\circ$ (i.e., for all normal operating conditions). (Note that in the literature, $k_1$ is cited as positive.) After carrying out the indicated algebra, the second condition reduces to

$$\cos 2\theta > -\frac{V_L^2 G_L}{P_0}$$

which is the same as obtained using the load model of equation (10.43) for the constant power part and presented earlier.

Case (ii) $k_4 = 0$

The elements of the state matrix are
VOLTAGE STABILITY

\[ a_{11} = -\frac{V_s V_L}{k_3 X} \sin \theta \quad a_{12} = -\frac{V_L}{k_3 X'} \]

\[ a_{21} = \frac{V_s V_L}{k_2 X} \left[ \cos \theta + \frac{k_2}{k_3} \sin \theta \right] \]

\[ a_{22} = \frac{V_L}{k_2 X'} \left[ \sin \theta + \frac{k_2}{k_3} - 2G_L X' \right] \]

For stability,

\[-(a_{11} + a_{22}) > 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} > 0\]

The second condition reduces to

\[ \cos 2\theta > -\frac{V_s^2 G_L}{P_0} \]

which is identical to that obtained for case (i). However, there seems to be some anomalies in satisfying the first condition. At a given load level, which may be well below the voltage stability limit obtained for case (i), the stability strongly depends on the relative values of \( k_1, k_2, \) and \( k_3 \).

However, by suitable choice of the parameter values, the small disturbance stability performance of the two dynamic load models described by equations (10.43) and (10.44) can be made identical. This is not surprising. In the absence of discontinuities (i.e., when \( \theta \) and \( V_L \) are differentiable), the two models are equivalent. For example, consider the case of the constant MVA load. Replace \( P_L \) and \( Q_L \) in equations (10.84) and (10.85) by \( V_s^2 G \) and \( V_s^2 B \), obtain the time derivatives of \( G \) and \( B \) in terms of \( \dot{\theta} \) and \( V'_L \), and substitute into (10.43). After carrying out the algebra and rearranging

\[ P_L = P(V) + k_1 \dot{\theta} + k_3 \dot{V}_L \]

\[ Q_L = Q(V) + k_3 \dot{\theta} + k_4 \dot{V}_L \]

where

\[ k_1 = -\frac{T_{LP} V_s}{V_L X} \cos \theta, \quad k_2 = \frac{T_{LP} G}{V_L} \]

\[ k_3 = \frac{T_{LQ} V_s}{V_L X} \sin \theta, \quad k_4 = \frac{T_{LQ} (1 + BX')}{V_L X'} \]

This explains why identical small disturbance performances were predicted by the two models. Note the negative sign of the expression for \( k_3 \). Note also that the model parameters are functions of system states. This may make the use of this model difficult in general simulation studies.

There is, however, a fundamental problem when using the Type-two model in a large disturbance analysis. In the network model employed in stability studies, the high frequency electrical transients are neglected and hence load bus voltages and angles are not state variables (i.e., they can have discontinuities). A model that employs time derivatives of these variables is therefore inconsistent with system modeling. The use of such a model would require adjusting the model parameter values during simulations whenever there is a discontinuity (e.g., a sudden disturbance). Serious computational difficulties can also be encountered.
Motor dynamic performance equations derived from a rigorous machine theory will contain time derivatives of voltages and angles. However, these are internal voltages and angles. Unlike motor terminal voltages and angles, these are true state variables.

One of the arguments presented in the literature in favor of the Type-two model is that it will serve as a generic model to predict large disturbance performance of any dynamic load including induction motor loads. However, the point is being made here that in many situations, the stability performance of induction motor loads determined from the basic dynamic model will be identical to that determined from a more detailed model, as shown earlier. When the basic model is not adequate, such as in some large disturbance situations, a comprehensive analysis employing detailed modeling of all pertinent system components would be needed. The use of the Type-two model would then certainly be questionable for the reason stated above.

### Corrective Measures for Voltage Stability

Concerns for voltage instability and collapse have prompted utilities to devise effective, efficient and economic solutions to the problem. Many utilities have either implemented or are actively considering the implementation of load shedding, capacitor switching, and under-load tap-changer (LTC) blocking, among other things, as emergency measures to combat voltage stability problems. However, confusion regarding the viability and effectiveness of these measures under voltage collapse conditions exists among utility engineers and system operators. Intuitively, one would not expect a decrease in the bus voltage magnitude when a capacitor is switched on, or some load is shed, at that bus, under *any* operating condition.\(^{(1)}\) The source of the confusion is the conventional steady-state analysis of voltage stability and misinterpretation of results obtained from the conventional power flow model. For example, sensitivity calculations using the formal power flow Jacobian will show both \(dV/dP\) and \(dV/dQ\) (or \(dV/dB\)) change sign as the operating point moves to the lower portion of the system \(PV\) curve. It is therefore natural to wonder whether load shedding or capacitor switching under such operating conditions would be a prudent thing to do.

The sensitivity results obtained from the conventional power flow Jacobian are not valid when applied to operation in the lower portion of the \(PV\) curve. This is due to the static, constant MVA load model generally used in the formulation of the power flow problem. While the static power flow solutions obtained from such a formulation would be mathematically correct, careful interpretation is necessary when using the sensitivity results. The difficulty clears up when one considers the behavior of constant MVA loads in real life. A constant MVA load is not a static load. As explained earlier, failure to consider the appropriate dynamics of the load may lead to erroneous and confusing conclusions. When the pertinent dynamics of the loads are taken into account, it can be shown that, while operating in the lower voltage region, system behavior following capacitor

---

\(^{(1)}\) There could be situations when shedding load would cause voltage collapse. However, this is related to the well known angle stability problem, not voltage stability. Consider, for example, the case where power is being transmitted from a remote generating source to a large system over a long transmission line, with some portion of the load tapped off near the source. If this load is lost, with the generation remaining constant, the transmission line would have to carry this additional load. If the line is not strong enough, there will be large angle difference between the generating source and the receiving system and voltages at intermediate points would be depressed. If there are loads connected at these points, some of that might be shed by undervoltage load shedding. This will aggravate the problem and could cause voltage collapse.
switching and/or load shedding is the opposite of what would be indicated from a steady-state analysis.

It is noted here that, depending on the composition and characteristic of the load, operation in the low voltage solution region can be viable, at least temporarily. It has been shown earlier that, if a substantial portion of the load is static, which may be the case in some power systems under cold weather conditions, stable operation in part of this region is possible, and no corrective measures may be necessary if the voltage level is acceptable. For loads with constant MVA (self restoring) characteristics, stable operation in this region is possible through the use of special controls, e.g., static var compensators (SVCs). Even without such controls, depending on the response speed of the load dynamics, there may be ample time available for corrective actions to avoid a voltage collapse.

Load shedding and capacitor switching can be useful and effective corrective measures to combat voltage stability problems in system operations. Load shedding will not normally be considered as an option in the planning against voltage collapse. However, no matter how well a power system is planned, occasionally situations will arise, such as disturbances which exceed planning and operating criteria, when load shedding may be the only available emergency corrective action that will avoid a complete system collapse. A compelling argument in favor of using load shedding as an emergency corrective measure [25] is that, since load will be lost anyway during abnormal voltage conditions, because of motor contactor dropout, discharge lighting extinction etc., it would be better to have the load shed under utility control. However, it is important to explore theoretically its viability under collapse conditions so as to identify any possible pitfall in the application of this last ditch effort.

A third measure often advocated under voltage collapse conditions is locking of taps or even tapping down of LTCs [27, 29], frequently utilized to maintain load voltages under normal operating conditions. While for many load types this would be effective in halting an impending collapse, indiscriminate LTC blocking, or tapping down may be counterproductive. A detailed and rigorous analysis is presented supporting this view.

**Load shedding**

There is some confusion as to what would actually happen if load shedding were attempted while operating in the lower portion of the system $PV$ curve. It has been shown that, in the absence of special controls, stable operation in this region is possible only when at least a part of the load is static. The operating points in the lower portion of $PV$ curves for such loads are illustrated by the intersections of the steady-state load characteristic and the system $PV$ curve in Figure 10.27. $A$ is the stable operating point, whereas point $A'$ is unstable. It is clear that, while operating at the stable operating point $A$, shedding some load will simply move the operating point upward along the system $PV$ curve, as illustrated by the intersection of the load characteristic, in dashed line, and the $PV$ curve.
For the purpose of illustration, consider a case when the load has constant power characteristic and the system is operating in the lower portion of the $PV$ curve. The operating point is shown at A, for a power level $P_o$ in Figure 10.28. Note that without special controls, the operating point cannot get into this region for constant power load, except temporarily. In other words, the system cannot be operated in the steady state in this region, for constant power load, unless special controls, such as SVCs, are utilized. This discussion is therefore mainly of academic interest.

Suppose now, an amount of load $\Delta P$ is shed. As the instantaneous load characteristic is constant impedance, the operating point will temporarily move to $B'$ following the load shedding. Since this is within the region of attraction of the stable equilibrium point $B$, corresponding to the new load $P$, the operating point will move to and settle at $B$. Note that the above scenario would be true irrespective of the magnitude of $\Delta P$.

In the numerical simulation of load shedding using the load model (10.43), it is only necessary to change the power set-point from $P_o$ to $P (= P_o - \Delta P)$. From (10.43) and from a consideration of the physical behavior of the system, it is clear that the system will settle at $B$ (Figure 10.28). Note, however, that along with the change in the power set-point, the system states $(G, V_L)$ could also be reinitialized, if desired, to reflect the instantaneous effect of load shedding as illustrated in Figure 10.28.
VOLTAGE STABILITY

We can use the same argument for the case shown in Figure 10.27, if the system is initially operating at \( A' \). The operating point immediately following the load shedding, given by the intersection of the instantaneous load characteristic (not shown in Figure 10.27, but similar to that shown in Figure 10.28) and the \( PV \) curve, will fall within the region of attraction of the stable equilibrium point corresponding to the new load. The operating point will therefore move and settle there.

Consider the case shown in Figure 10.29, where the system is initially operating at load level \( P_o \), which is greater than the maximum load capability of the post-disturbance system, \( P_{\text{max}} \). It is clear that, for constant power load, the minimum load needed to be shed in order for the post-disturbance system to have an equilibrium point is \((P_o - P_{\text{max}})\). However, in order to keep the load shedding requirement to a minimum, load must be shed promptly. The reason is as follows:

Immediately following the disturbance the system state moves from \( A \) to \( A' \). If the load tries to maintain constant power characteristic, the system state would start moving down the post-disturbance system \( PV \) curve on the way to voltage collapse. As the system state immediately following the load shedding must be within the region of attraction of the final stable equilibrium point, the need for a prompt load shedding is clear. A delayed shedding would require shedding of load in amount much greater than \((P_o - P_{\text{max}})\), in order to bring the system state within the region of attraction. The delay that can be tolerated would depend on the speed of the mechanism (the value of \( T_L \) in (10.43)) that restores the load to constant power.

![Fig. 10.29 Illustration of the minimum load needed to be shed as an emergency control measure when the pre-disturbance (constant MVA) load is greater than the maximum load supplying capability of the post-disturbance system.](image)

For loads with fast response characteristics, e.g., motor loads, the voltage will drop at a much faster rate than would be the case, for example, if the load were static and restored to constant MVA by LTC action. The rate of drop of voltage would, therefore, serve as an indicator for an immediate need for load shedding.

It is also clear that the load shedding requirement is less stringent when a part of the load is static. If a large portion of the load is static with overall load characteristic similar to that shown in Figure 10.27, as opposed to a vertical line for a constant power load, load shedding may not be necessary, except to bring back the voltage to acceptable level, if it has gone down too low. This is because a stable equilibrium point may already exist in the post-disturbance system, although the resulting
operating point may be at a much lower voltage with consequent reduction in the load power delivered, which may or may not be acceptable.

Several points need to be kept in mind while considering implementation of a viable load shedding program.

1. There is a minimum amount of load that needs to be shed in order to restore stability.
2. Unless prompt load shedding is initiated, more load would need to be shed than necessary. This might lead to the opposite problem of high voltage.
3. Although low voltage would be a good indicator for initiating load shedding, low voltage alone need not be used as a reason for load shedding, unless the low voltage itself is unacceptable. Any undervoltage load shedding scheme should also take into account the rate of drop of voltage in order to ensure the need for an immediate load shedding. A fast rate of drop of voltage would indicate voltage instability caused by stiff load, e.g., motor load. This would require prompt shedding of such loads.

**Capacitor switching**

In the previous example, instead of load shedding, additional capacitors, if available, could be employed in order to restore a stable equilibrium point. This is illustrated in Figure 10.30. Immediately following the disturbance the system state moves to A’. If a sufficient quantity of capacitors are promptly switched on at or near the load bus, so that a stable equilibrium state is possible, and the initial state following the capacitor insertion is within the region of attraction of this stable equilibrium state, stability can be maintained. In the illustration of Figure 10.30, the system state moves from A’ to B’ following capacitor insertion, then moves to B, the final stable equilibrium point.

![Fig. 10.30 Restoration of stable equilibrium point following a disturbance by capacitor switching.](image)

The necessity of prompt insertion of capacitors for maintaining stability should be clear from the above illustration. The switching must be accomplished before the system state moves too far down the post-disturbance PV curve. Beyond a certain point, no amount of capacitor switching will restore stability. Again, the capacitor switching speed requirement is not as stringent when part of the load is static. This is due to the shape of the overall steady-state load characteristic.
Although load supplying capability can be greatly increased by providing reactive support by capacitors at or near the load bus, beyond a certain power level, continuous adjustment of reactive support as provided by SVCs or synchronous condensers would be required, if voltages are to be maintained at acceptable levels. Maintaining normal voltage with reactive support at high power transfer may mean operating in the lower region of the PV curve. As will be shown later, stable operation in this region can be maintained by fast and continuous adjustment of reactive support as provided by SVCs.

A conceptual understanding of the issues involved can be obtained by referring to Figure 10.31. The system PV curves shown correspond to various amounts of reactive support at the load end. Suppose the initial operating point is at A, corresponding to load \( P_o \) and reactive support \( B_2 \). With continuous reactive support, adjusted to maintain constant load voltage, the operating point would move approximately along the line OO’ as the load power is varied. (Stable operation along OO’ using SVC will be discussed in the next section.) The effect of providing reactive support by means of discrete capacitors will now be investigated.

Suppose the load is increased to \( P_1 \). From Figure 10.31, it is clear that without additional reactive support this load cannot be maintained and, therefore, voltage collapse will occur. Now suppose an additional reactive support is provided and the PV curve corresponding to \( B_3 \) applies. The operating point will temporarily move to \( B’ \) (as the instantaneous behavior of all loads is static, the instantaneous load characteristic following each load change would be as shown by the dashed lines). As B’ is within the region of attraction of the stable equilibrium point B, the operating point will eventually move to B. Note, however, that the voltage may be unacceptably high.

![Fig. 10.31 Illustration of the issues involved in increasing load supplying capability using reactive support by switched capacitors.](image)

Following the same argument, it is clear that if the additional reactive support were provided at the original load level \( P_o \), the operating point would have moved to \( A’ \), irrespective of whether the initial operating point were on the upper portion (A), or the lower portion of the initial PV curve. This is because the intersection of the instantaneous load characteristic and the PV curve.
corresponding to $B_3$ would fall within the region of attraction of the stable equilibrium point $A'$. A steady-state analysis, on the other hand, will suggest that the operating point would move to $A''$, if the initial operating point were on the lower portion of the $PV$ curve.

If, instead, the load is increased to $P_2$, and sufficient reactive support for an equilibrium point to exist in the steady state is provided, say $B_4$, the operating point will momentarily move to $C'$. As $C'$ is outside the region of attraction of the stable operating point $C$, voltage collapse will occur. In order to maintain stability, additional reactive support would have to be provided, and the operating point will settle at an even higher voltage (in the example, the operating point settles at $D$ corresponding to a reactive support of $B_5$).

If the reactive support were increased in anticipation of a load increase, the reactive requirement for maintaining stable operation would be lower. In the example provided, a stable operating point $C$, corresponding to load $P_2$ could be reached with reactive support $B_4$, if the increase in reactive support preceded the load increase. However, the problem of high voltage will still be present.

As before, it is not difficult to see that the situation is not so severe when a portion of the total load is static.

**Extending Voltage Stability Limit by SVC**

If high voltage caused by simple shunt capacitors is not acceptable, or when fast control of load voltage is required, the use of controlled reactive support such as provided by an SVC may be necessary. Stable operation in the lower portion of the $PV$ curve for constant MVA load (self-restoring load) can be satisfactorily explained by simultaneously taking into consideration the relevant dynamics of the load and the SVC control mechanism. Note that, for static load, such as constant impedance, SVC is not necessary for maintaining voltage stability, although it will certainly help in precise voltage control.

---

![Diagram](image)

**Fig. 10.32** A power system with constant sending-end voltage supplying a load whose voltage is maintained constant by an SVC.

We now provide a simple but rigorous explanation of the mechanism of the extension of voltage stability limit due to SVC. Consider the system shown in Figure 10.32, supplying a unity power factor load whose voltage is being controlled by an SVC. An SVC acts by increasing capacitive susceptance as the voltage drops from a set value and vice versa. For the present purpose, the control logic of the SVC may be described by the simple first order delay model.
\[ T_Q \dot{B} = V_{\text{ref}} - V_L \]  

(10.86)

where \( B \) is the SVC susceptance and \( T_Q \) is a time constant. The load model is given by (10.43). The power balance equations are:

\[ P = V_L^2 G = \frac{V_s V_L}{X} \sin \theta \]  

(10.87)

\[ Q = 0 = \frac{V_s V_L}{X} \cos \theta - \frac{V^2}{X} + V_L^2 B \]  

(10.88)

After linearizing equations (10.43), (10.86) – (10.88), and eliminating the non-state variables, the state-space model is obtained as

\[
\begin{bmatrix}
\Delta \dot{B} \\
\Delta \dot{G}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta B \\
\Delta G
\end{bmatrix}
\]

where

\[
a_{11} = -\frac{V_L}{2T_Q G} \sin 2\theta, \quad a_{12} = \frac{V_L}{T_Q G} \sin^2 \theta
\]

\[
a_{21} = -\frac{V^2}{T_L} \sin 2\theta, \quad a_{22} = -\frac{V_L}{T_L} \cos 2\theta
\]

For stability, \(-(a_{11} + a_{22}) > 0, \quad a_{11}a_{22} = a_{12}a_{21} > 0\). After some algebraic manipulation and noting from (10.87) and (10.88) that \(\tan \theta = GX/(1-BX)\), the following stability condition results:

\[
1 - \tan^2 \theta > \frac{T_L}{T_QV_L} \frac{X}{1-BX}
\]  

(10.89)

Equation (10.89) shows that, when \(T_Q \ll T_L\), which would be true for most load types, the voltage stability limit can extend to \(\theta\) approaching 90°, i.e., well into the lower portion of the \(PV\) curve. (Note that for unity power factor load, \(\theta = 45°\) corresponds to the maximum power point on the \(PV\) curve, the so-called static bifurcation point; \(\theta > 45°\) corresponds to operation in the lower portion of the curve. Also, beyond \(\theta = 45°\), the reactive requirement increases rapidly.) On the other hand if \(T_L = 0\), i.e., if the load adjusts to constant power instantaneously, the stability limit occurs at \(\theta = 45°\), implying that no improvement in the voltage stability limit over that obtainable from static shunt reactive support is possible. It may be of interest to note that if, in the above analysis, a constant power static load model is used, the result will correspond to that for the dynamic load case with \(T_L = 0\).

The above analysis explains how an SVC can maintain stable operation in the lower portion of the \(PV\) curve for real life constant power load. With an SVC, stable operation can be realized as the load is varied at a given voltage level, for example, along the line \(OO'\) as illustrated in Figure 10.31, or as the voltage set-point is varied at a given load power level, or for any combination thereof.

The operation along the horizontal line \(OO'\) in Figure 10.31 (in actual operation the SVC control may not act like a simple integrator as implied by (10.86) and the line \(OO'\) may have a small droop)
is easy to visualize. For example, the instantaneous effect of a load increase is an increase in the load admittance, resulting in a drop in voltage. The SVC responds by increasing the capacitive susceptance until voltage is restored and the desired load level is reached.

In order to visualize stable operation at a given load level following a change in the voltage set-point, consider a lowering of the voltage set-point. The SVC will initially decrease the capacitive susceptance (cf. equation (10.86)). This will result in a decrease in voltage and hence in load, as the instantaneous characteristic of the load is static. As the load tries to restore to the original level (cf. equation (10.43)), the voltage will decrease further and the SVC will compensate by increasing the capacitive susceptance. A stable operating point at the lower voltage level would be reached by an eventual increase in the capacitive susceptance as long as $T_Q < T_L$.

The above scenario is illustrated by the simulation results shown in Figure 10.33. Referring to the system shown in Figure 10.32, the system parameter values and initial conditions are as follows: $X = 0.5$, $T_Q = 0.1$, $T_L = 1.0$, $V_S = 1.0$, $V_L = 1.0$, $P = 1.6$, $Q = 0.0$. The initial operating point values are computed as $\theta = 53.13^\circ$ and $B = 0.8$. The initial operating point is thus in the lower portion of the $PV$ curve. Response curves shown in Figure 10.33 correspond to a change in the voltage set-point, $V_{ref}$, from 1.0 to 0.95. Note the behavior of the SVC output ($B$). Its initial response is in a direction opposite to its final settling point.

Note that when the increase in reactive support corresponds to a decrease in voltage, the operation is actually stable, contrary to the popular (static) definition of voltage instability.

![Figure 10.33](image)

Fig. 10.33 Response curves following a lowering of the voltage set-point of the SVC while operating in the lower portion of the $PV$ curve.

When the receiving-end voltage is being controlled by an SVC, and the SVC capacitive limit is encountered, it would behave like regular capacitors. If the operation is on the upper portion of the $PV$ curve, there would be no stability consequence. However, the stability consequence of the SVC
reaching its capacitive limit when the system is operating in the lower portion of the $PV$ curve is of considerable importance. Consider two scenarios: one in which the limit is reached while the SVC is controlling the voltage in response to load increase, and the other where the SVC is trying to restore voltage following a contingency. In the first scenario, referring to Figure 10.31, assume that the desired operation is on the line $OO'$ as the load is increased from $P_0$ to $P_2$. Now assume that the SVC limit is reached so that, at limit, the $PV$ curve corresponding to $B_4$ applies. At the time the SVC limit is reached, the system state is at $C'$, the intersection of the instantaneous load characteristic and the system $PV$ curve corresponding to $B_4$. As it is outside the region of attraction of the stable equilibrium point of the final system, voltage collapse would occur if the load tries to reach the constant power level $P_2$. Note that the voltage would collapse even though a stable operating point exists in the final system. It is not difficult to visualize that stable operation would be maintained if the load had a substantial portion of static component.

In the second scenario, the SVC will try to restore the voltage, and will succeed if sufficient reactive were available. If in the process of restoring voltage the limit is encountered, it is clear from the argument presented earlier, that the system state at the time the limit is reached (the intersection of the instantaneous load characteristic and the post-disturbance $PV$ curve at the full SVC reactive output) would be outside the region of attraction of the stable equilibrium point of the final system, so that voltage collapse would take place. In the above scenario it was assumed that a stable equilibrium state is possible in the post-disturbance system; however, an equilibrium point would probably not exist if the load is of constant power type. As before, the situation would be less stringent if static load comprised a portion of the total load.

**LTC Operation and Voltage Stability**

LTC operation can have either a beneficial or an adverse effect on voltage stability, depending on the LTC's location with respect to the load point and the type of load. Blocking of LTC is often advocated under voltage collapse situations [27, 29]. The argument for this is that normal LTC operation tends to render any load constant MVA which, in turn, would aggravate any existing voltage stability problem. Indiscriminate blocking of LTC, (or tapping down), without regard to the type of load or the location of the LTC in relation to the load bus may, however, be counter-productive.

![Fig. 10.34 A simple power system with LTC located away from the load bus and controlling voltage locally: (a) original circuit, (b) Thevenin equivalent. The phasor diagram applies to unity pf load.](image)

The issues will be clarified by first presenting a basic analysis of voltage stability as affected by LTC operation. For simplicity, consider a system as shown in Figure 10.34a, supplying a unity...
power factor load whose dynamic behavior is given by (10.43). For generality, assume the LTC
located away from the load bus and controlling the voltage locally, as shown in the figure. In this
way, the conclusions drawn from the analysis will also be applicable to induction motor loads, as
the circuit of Figure 10.34a, along with the load model (10.43), approximately represents a power
system supplying an induction motor load through a transmission line, with an LTC controlling the
load voltage. For clarity, shunt support is not included in the analysis. Its effect can be included by
simply modifying the source voltage and the reactance between the source voltage and the shunt
support point by a factor $1/(1-BX)$, where $B$ is the shunt susceptance and $X$ is the reactance between
the source voltage and the shunt point.

The analysis can be greatly simplified if the original circuit of Figure 10.34a is replaced by its
Thevenin equivalent shown in Figure 10.34b. The tap changer dynamics will be represented by a
continuous model similar to that given in Reference 31, and as shown by the following equation

$$T \dot{n} = V_{ref} - V_2$$

(10.90)

where $n$ is the tap position and $T$ is the time delay associated with tap changing. The load model is
given by (10.43). The power balance equations are:

$$P = V_L^2 G = \frac{nV_L^2V_L}{n^2X_1 + X_2} \sin \theta$$

(10.91)

$$Q = 0 = \frac{nV_L^2V_L}{n^2X_1 + X_2} \cos \theta - \frac{V_L^2}{n^2X_1 + X_2}$$

(10.92)

Also, from the phasor diagram,

$$V_2^2 = V_L^2 (1 + G^2 X_2^2)$$

(10.93)

After linearizing equations (10.43), (10.90) – (10.93), and eliminating the non-state variables, the
state-space model is obtained as

$$\begin{bmatrix} \Delta \dot{n} \\ \Delta \dot{G} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta n \\ \Delta G \end{bmatrix}$$

where

$$a_{11} = -\frac{V_L^2}{T_n} \frac{n^2X_1 \cos 2\theta + X_2}{n^2X_1 + X_2}, \quad a_{12} = \frac{V_L^2}{TV_2 G} (G^2 X_2^2 \cos 2\theta - \sin^2 \theta)$$

$$a_{21} = -\frac{2V_2 G}{T_L n} \frac{n^2X_1 \cos 2\theta + X_2}{n^2X_1 + X_2}, \quad a_{22} = -\frac{V_L^2}{T_L} \cos 2\theta$$

For stability, $-(a_{11} + a_{22}) > 0$, $a_{11}a_{22} - a_{12}a_{21} > 0$. After some algebraic manipulation the
stability conditions reduce to

$$\cos 2\theta > -\frac{X_2}{n^2X_1 + \frac{T_n V_2^2}{T_L V_2} (n^2X_1 + X_2)} \quad \text{(10.94)}$$

$$1 > G X_2 \quad \text{(10.95)}$$
The condition given by (10.95) is trivial; it simply refers to the voltage stability limit of the part of the system with $V_2$ as the constant sending-end voltage. Therefore, it is necessary to concentrate only on the condition given by (10.94). Note that for $X_2 = 0$, the stability condition reduces to $\cos 2\theta > 0$, i.e., $\theta < 45^\circ$, irrespective of the type of load. The stability limit corresponds to the maximum power point. This is also the stability condition for constant power load without the LTC. LTC, therefore, has no effect on voltage stability limit for constant power load when it is located at the load bus. If the load exceeds this limit the voltage will collapse. Blocking of tap changing will help avoid an impending collapse when the load is static. Note that certain apparently static loads, such as thermostatically controlled electric space heating, tend to be dynamic, self-restoring in the longer term. Tap changer blocking is not going to be very effective for such loads in the longer term, although it will buy some time to initiate other measures.

For constant power load, the effect of discrete tap changing when the load is at or close to the limit may be of some interest. Depending on the margin left and the step size, raising the tap setting may bring on instability. This is illustrated in Figure (10.35a). The initial operating point is at A, on the $PV$ curve corresponding to the lower tap ratio $n_1$. $A'$ is the (stable) equilibrium point for this load level corresponding to the higher tap ratio $n_2$. In this illustration the operating point immediately following the tap changing, given by the intersection of the instantaneous load characteristic and the $PV$ curve corresponding to the tap ratio $n_2$, falls outside the region of attraction of $A'$, resulting in instability.

For $X_2 \neq 0$, consider the following cases:

Case (i): Static load, $T_L = \infty$, and (10.94) reduces to

$$\cos 2\theta > -\frac{X_2}{n^2 X_1}$$

Stability limit extends beyond $\theta = 45^\circ$, i.e., into the lower portion of the $PV$ curve. This would be the case if there is a substantial length of feeder circuit between the tap-changer and the load.

Case (ii): Constant power dynamic load with response time much faster than the LTC response time, $T >> T_L$. The stability condition reduces to $\cos 2\theta > 0$, i.e., $\theta < 45^\circ$. The stable operation is therefore confined to the upper portion of the $PV$ curve.
Fig. 10.35 PV curves for two values of tap ratio: (a) LTC connected directly to the load; (b) with an impedance between LTC and load point.

The maximum power that can be delivered is given by

$$P_{\text{max}} = \frac{(nV_s)^2}{2(n^2X_1 + X_2)} = \frac{V_s^2}{2(X_1 + X_2/n^2)}$$

(10.97)

When $X_2 = 0$, the maximum power delivered is independent of the tap position. With the tap-changer away from the load end, such as when there is a length of feeder circuit between the tap-changer and the load, the maximum power is increased by raising tap position. (Recall that for constant power load, an LTC does not affect the angle at which voltage stability limit occurs.) The implication is that, while operating close to the limit, tapping down can have a detrimental effect. This is illustrated in Figure 10.35b, which shows the PV curves corresponding to two tap ratios, $n_1$ and $n_2$ ($n_2 > n_1$), when there is some impedance between the LTC and the load delivery point. Suppose the system is delivering a constant power load $P$, at tap ratio $n_2$. The operating point is at A. As the power level exceeds the maximum power that can be delivered at tap ratio $n_1$, tapping down from $n_2$ to $n_1$ can initiate a collapse. The same argument can be made against LTC blocking when there is a likelihood of a load increase beyond the maximum power that can be delivered at the current tap setting.

Even for static load and operating in the lower portion of the PV curve, blocking or tapping down can be counterproductive. This is illustrated in Figure 10.35b for the load characteristic ab. It is evident that while operating at b, lowering of tap will cause both the power delivered and the load voltage to go down. Similarly, while operating at a, raising of tap is preferable to locking.

As the circuit of Figure 10.34a approximately represents an induction motor whose terminal voltage is being controlled by an LTC, it follows that tapping up near stability limit can be beneficial, especially if there is some capacitive reactive support on the load side of the LTC.

Tap-changer dynamics act much slower than the dynamics of loads with inherently constant power characteristics, such as induction motors. LTCs cannot, therefore, play any role in modifying the dynamic voltage stability behavior due to such loads. If the induction motor has already become unstable, i.e., it is in the unstable region of the torque-slip curve, raising or lowering of tap is not likely to make much difference, considering the response speed of the motor and the limited range
in the tap settings. The motor is going to stall, unless some mechanical load is shed somehow. Similarly, in the stable region, tap changing will not affect the stability of the operating point significantly. Raising the tap position would, however, raise the peak of the torque-slip curve somewhat, thereby increasing the stability margin. The opposite will hold if the tap position is lowered. If the motor is operating near the stability limit, tapping down may have a destabilizing effect. Actually, near a stability limit, the performance following a tap change would depend on the step size, location of other motors in the system as well as the motor and other system parameters. Some specific cases, where tapping up near a stability limit can cause problem, are discussed in Reference 32. If motor stability is in question, a detailed dynamic analysis, employing detailed representations of the motors and voltage control devices should be performed before deciding upon the appropriate tap changing action.

In a typical utility system much of the loads at certain locations may be static, such as constant impedance, which are rendered constant MVA by LTCs. In such situations, blocking of tap changing or even tapping down under low voltage conditions, especially following contingencies, would certainly help. With LTCs blocked, static loads will remain static, and no voltage instability can occur. It has been shown that when a substantial portion of the load is static, the pre-contingency load can be greater than the post-contingency power capability of the system, and still maintain stability following the contingency.

**Voltage Stability Studies in Actual Power Systems**

Voltage stability is load driven. If the overall load characteristic is such that voltage stability cannot occur, one has to contend with the voltage collapse problem due to other causes, e.g., parts of the system exceeding angle stability limits, low voltage caused by heavy network loading and insufficient reactive support, and/or generators reaching reactive limits. These are steady-state problems and can be handled by conventional power flow programs. This problem of voltage and reactive power control has been well understood by system planners and operators for as long as power systems have existed and discussed in details in the literature [1, 2], although in recent years this has been confused with voltage instability. True voltage instability can be caused only when the bulk of the load is fast response load of self-restoring type, e.g., induction motor load. (Note that synchronous motors, like synchronous generators, can cause only angle instability, and occasional voltage collapse resulting from that instability.) From the analyses presented throughout this chapter it should be clear that, when the motor mechanical torque is speed dependent, voltage stability issues are mitigated. (Actually, if the motor mechanical torque is directly proportional to speed, or any power of speed, voltage instability cannot occur; however, voltage collapse, defined as low voltage when the motor cannot deliver its designed performance, can remain an issue.)

The only way to study and analyze true voltage stability problems in real power systems is to use a dynamic performance analysis program representing the detailed voltage control and load dynamics. Any commercially available program designed for dynamic performance analysis can be used for this purpose as long as the appropriate load dynamics are represented (all these programs come with adequate dynamic load models -- it’s a matter of choosing the right model for the problem at hand). The specialized computer programs designed for “voltage stability” studies are based on steady-state or quasi steady-state formulation and as such they are not suitable for voltage stability analysis, although they might appear to be so. When there is no true voltage stability problem because of the overall system load characteristic, these programs may
be a more efficient alternative to the conventional power flow program. In situations where voltage instability is a real possibility these programs are not recommended.

A Typical Voltage Collapse Scenario

In a modern interconnected power system, a total system collapse is a relatively rare occurrence. A study of the world’s major voltage collapse incidents reveals certain similarities in the system operating conditions existing prior to the collapse, and the subsequent sequence of events that led to the collapse. The initiating event may be a large disturbance, e.g., the loss of a heavily loaded line and/or a large generating unit. This kind of contingency is considered in system planning and in most cases this would not cause serious problem. Occasionally, however, the system may already be overloaded beyond planning limit, and/or another line or equipment is outaged, either inadvertently or due to protective action. This drives the system beyond its loading limit and results in depressed voltages at the load centers. If the load is predominantly dynamic, e.g., motor load with fast recovery characteristics, voltage instability and collapse would follow quickly.

A more likely scenario might, however, be the following: In a typical utility system the composite load characteristic being voltage sensitive and/or slow recovery type, an immediate voltage instability and collapse does not occur. Due to the depressed voltage there is considerable load relief initially. In parts of the system some load might trip by undervoltage protection due to excessively low voltage. At this time the system’s additional reactive demand (caused by excessive reactive losses due to system overload) is being met by the temporary reactive support from the remaining generators utilizing their short-term overload capability.

Due to the load relief caused by depressed voltage there is now excess generation and one or more generators trip from overspeed. As reactive supports from these generators are lost the voltages are depressed further resulting in further overloading of transmission lines, some of which might start to trip by protective action. By this time some of the loads attempt to restore to their nominal level.

By now the remaining generators’ short-term (overload) reactive capability is exhausted and the overexcitation limiters start to bring the excitation down to the steady-state limit. There is now serious reactive imbalance in the system. This causes widespread voltage decline and frequency swing, resulting in tripping of more generators and loads. Soon the entire system collapses. The collapse takes place over a period of several to many minutes. Note that speed governors, AGC and other power plant controls which would normally respond to generation load imbalance could not be very effective due to the severity of the problem, and, in fact, might have aggravated the problem.

Notice that in the scenario presented there was no rotor angle instability or voltage instability in the conventional sense. In fact, if a simulation of the initiating events were carried out using a conventional transient stability program, everything would probably have looked normal in the simulated time span of a few seconds. There was no transient or oscillatory instability. Voltage did not collapse because of voltage instability -- as the events progressed, voltages went down in steps following each event and stayed there until the next one. The system collapsed because of the continued imbalance in generation and load, accompanied by the depletion of reactive reserve. This was a case of overall system instability and collapse.
If the final outcome were anticipated at the initial stage, from the overall system condition prevailing at the beginning of the disturbance, well before the onset of the collapse, the system might have been saved by operator action, by promptly initiating appropriate amount of load shedding at the appropriate locations. The initial stage may not, however, indicate the need for such actions.

References


APPENDIX A
REVIEW OF MATRICES

The set of linear equations

\[a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = y_1\]
\[a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = y_2\]
\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = y_m
\]

(A.1)

constitutes a set of relationships between the variables \(x_1, x_2, \cdots, x_n\), and the variables \(y_1, y_2, \cdots, y_m\). This relationship, or linear transformation of the \(x\) variables into the \(y\) variables is completely characterized by the ordered array of the coefficients \(a_{ij}\). If this ordered array is denoted by \(A\), and written as

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

(A.2)

then the set of linear equations can be written as \(Ax = y\), by a suitable definition of the product \(Ax\).

A matrix is a rectangular array such as shown in equation (A.2) that obeys certain rules of addition, subtraction, multiplication, and equality.

The elements of the matrix, \(a_{11}, a_{12}, \cdots, a_{ij}\), are written with a double subscript notation. The first subscript indicates the row where the element appears in the array, and the second subscript indicates the column. The elements may be real or complex, or functions of specified variables. A matrix with \(m\) rows and \(n\) columns is called an \(m \times n\) (\(m\) by \(n\)) matrix or is said to be of order \((m, n)\). For a square matrix \((m = n)\) the matrix is of order \(n\). It is common to use bold-faced letters to denote matrices.

**Types of matrices**

Column matrix: An \(m \times 1\) matrix is called a column matrix or a column vector, such as

\[
a = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{bmatrix}
\]

Row matrix: A matrix containing a single row of elements, such as a \(1 \times n\) matrix, is called a row matrix or a row vector.
Diagonal matrix: The principal diagonal of a square matrix consists of the elements \( a_{ii} \). A diagonal matrix is a square matrix whose off-diagonal elements are zero. Denoting the diagonal matrix by \( D \),

\[
D = \begin{bmatrix}
d_{11} & & \\
& d_{22} & \\
& & \ddots \\
& & & d_{nn}
\end{bmatrix}
\]

Unit matrix: A unit matrix is a diagonal matrix whose principal diagonal elements are equal to unity. It is often denoted by \( I \).

\[
I = \begin{bmatrix}
1 & & \\
& 1 & \\
& & \ddots \\
& & & 1
\end{bmatrix}
\]

Null matrix: A matrix which has all of its elements identically equal to zero is called a zero or null matrix.

Transpose of a matrix: Transpose of a matrix \( A \) is obtained by interchanging the rows and columns of \( A \). The transpose of a matrix \( A \) is denoted by \( A' \) (or \( A^T \)). If \( A \) is an \( m \times n \) matrix, then \( A' \) is an \( n \times m \) matrix.

Symmetric matrix: A matrix is said to be symmetric if it is equal to its transpose, i.e., if \( A = A' \).

Skew-symmetric matrix: A matrix is said to be skew-symmetric if \( A = -A' \). This, of course, implies that the elements on the principal diagonal are identically zero.

Conjugate matrix: If the elements of the matrix \( A \) are complex, then the conjugate matrix \( B \) has elements which are complex conjugates of the elements of \( A \). This is written as \( B = A^* \).

**Addition of matrices**

If two matrices \( A \) and \( B \) are both of order \((m, n)\), then the sum of these two matrices is the matrix \( C \), \( C = A + B \), where the elements of \( C \) are defined by

\[
c_{ij} = a_{ij} + b_{ij} \quad (A.3)
\]

Addition of matrices is commutative and associative, i.e.,

\[
A + B = B + A \quad \text{(commutative)}
\]

\[
A + (B + C) = (A + B) + C \quad \text{(associative)}
\]

**Subtraction of matrices**

The difference of two matrices \( A \) and \( B \), both of order \((m, n)\), is the matrix \( D \), \( D = A - B \), where the elements of \( D \) are defined by

\[
d_{ij} = a_{ij} - b_{ij} \quad (A.4)
\]
**Equality of matrices**

Two matrices $A$ and $B$, which are of equal order, are equal if their corresponding elements are equal. Thus $A = B$, if and only if $a_{ij} = b_{ij}$.

**Multiplication of matrices**

Consider the set of equations

$$
\begin{align*}
y_1 &= a_{11} x_1 + a_{12} x_2 \\
y_2 &= a_{21} x_1 + a_{22} x_2
\end{align*}
$$

(A.5)

Equation (A.5) can be visualized as the matrix $A$ transforming the vector $x$ into the vector $y$. This transformation can be written in the form

$$
y = Ax
$$

(A.6)

where

$$
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Assume now that the vector $x$ is formed by the linear transformation

$$
\begin{align*}
x_1 &= b_{11} z_1 + b_{12} z_2 \\
x_2 &= b_{21} z_1 + b_{22} z_2
\end{align*}
$$

(A.7)

which can be written in the form

$$
x = Bz
$$

(A.8)

The transformation from the column vector $z$ to the column vector $y$ can be written in matrix notation as

$$
y = ABz
$$

(A.9)

The product $AB$ can be viewed as the matrix $C$ where

$$
C = AB
$$

(A.10)

A typical element $c_{ij}$ of $C$ is given by

$$
c_{ij} = \sum_{k=1}^{2} a_{ik} b_{kj}
$$

(A.11)

**Problem**

Verify equation (A.11) by obtaining the relationship between the vectors $y$ and $z$ by substituting (A.7) into (A.5)

In order for the product to be defined, the number of columns of $A$ must equal the number of rows of $B$. Proceeding to the general case, the product of two matrices, $A$ ($m \times n$) and $B$ ($n \times p$), is defined in terms of the typical element of the product $C$ as

$$
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}
$$

(A.12)

Clearly, the product $C$ is an $m \times p$ matrix. In general, multiplication is not commutative, i.e.,

$$
AB \neq BA
$$
Matrix multiplication is associative and distributive.
\[
A(BC) = (AB)C \quad \text{(associative)}
\]
\[
A(B + C) = AB + AC \quad \text{(distributive)}
\]

**Scalar multiplication**

Pre-multiplication or post-multiplication of a matrix by a scalar multiplier \( k \) multiplies each element of the matrix by \( k \).

**Multiplication of transpose matrices**

The product of two transpose matrices, \( B' \) and \( A' \), is equal to the transpose of the product of the original two matrices \( B \) and \( A \), taken in reverse order, i.e.,
\[
B'A' = (AB)'
\]

**Problem**

Verify equation (A.13).

**Multiplication by a diagonal matrix**

Post-multiplication of a matrix \( A \) by the diagonal matrix \( D \) is equivalent to an operation on the columns of \( A \). Pre-multiplication of a matrix \( A \) by the diagonal matrix \( D \) is an operation on the rows of \( A \).

**Problem**

Verify the above two statements.

Obviously, pre-multiplication or post-multiplication by the unit matrix \( I \) leaves the matrix unchanged, i.e., \( IA = AI = A \).

**Partitioning of matrices**

Sometimes it is convenient to partition matrices. For example, by drawing the vertical and horizontal dotted lines in the \( 3 \times 3 \) matrix \( A \), a \( 2 \times 2 \) matrix can be written using the submatrices \( A_1, A_2, A_3, \) and \( A_4 \).

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
  A_1 & A_2 \\
  A_3 & A_4
\end{bmatrix}
\]

Assuming that the matrix \( B \), which is also of order (3, 3) is partitioned in the same manner, yielding

\[
B = \begin{bmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{bmatrix} = \begin{bmatrix}
  B_1 & B_2 \\
  B_3 & B_4
\end{bmatrix}
\]
REVIEW OF MATRICES

\[
A + B = \begin{bmatrix}
A_1 + B_1 & A_2 + B_2 \\
A_3 + B_3 & A_4 + B_4
\end{bmatrix}
\]

and

\[
A \cdot B = \begin{bmatrix}
(A_1 B_1 + A_2 B_3) & (A_1 B_2 + A_2 B_4) \\
(A_3 B_1 + A_4 B_3) & (A_3 B_2 + A_4 B_4)
\end{bmatrix}
\]

In general, the product of two matrices can be expressed in terms of the submatrices, only if the partitioning produces submatrices which are conformable. The grouping of columns in \( A \) must be equal to the grouping of rows in \( B \).

**Differentiation of a matrix**

Let \( A(t) \) be an \( m \times n \) matrix whose elements \( a_{ij}(t) \) are differentiable functions of the scalar variable \( t \). The derivative of \( A(t) \) with respect to the variable \( t \) is defined as

\[
\frac{d}{dt}[A(t)] = \dot{A}(t) = \begin{bmatrix}
\frac{d}{dt} a_{11}(t) & \frac{d}{dt} a_{12}(t) & \cdots & \frac{d}{dt} a_{1n}(t) \\
\frac{d}{dt} a_{21}(t) & \frac{d}{dt} a_{22}(t) & \cdots & \frac{d}{dt} a_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d}{dt} a_{m1}(t) & \frac{d}{dt} a_{m2}(t) & \cdots & \frac{d}{dt} a_{mn}(t)
\end{bmatrix}
\]

(A.14)

From above, it is evident that

\[
\frac{d}{dt}[A(t) + B(t)] = \dot{A}(t) + \dot{B}(t)
\]

(A.15)

The derivative of a matrix product is formed in the same manner as the derivative of a scalar product, with the exception that the order of the product must be preserved. Examples:

\[
\frac{d}{dt}[A(t)B(t)] = \dot{A}(t)B(t) + A(t)\dot{B}(t)
\]

(A.16)

\[
\frac{d}{dt}[A^2(t)] = \dot{A}(t)A^2(t) + A(t)\dot{A}(t) + A^2(t)\dot{A}(t)
\]

(A.17)

**Integration of a matrix**

The integral of a matrix is defined in the same way as the derivative of a matrix. Thus

\[
\int A(t) \, dt = \begin{bmatrix}
\int a_{11}(t) \, dt & \int a_{12}(t) \, dt & \cdots & \int a_{1n}(t) \, dt \\
\int a_{21}(t) \, dt & \int a_{22}(t) \, dt & \cdots & \int a_{2n}(t) \, dt \\
\vdots & \vdots & \ddots & \vdots \\
\int a_{m1}(t) \, dt & \int a_{m2}(t) \, dt & \cdots & \int a_{mn}(t) \, dt
\end{bmatrix}
\]

(A.18)

**Determinant**

The determinant of a square matrix \( A \) is denoted by enclosing the elements of the matrix \( A \) within vertical bars. For example, for a \( 2 \times 2 \) matrix \( A \),
REVIEW OF MATRICES

\[
\text{det } A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\]  \hspace{1cm} (A.19)

If the determinant of \( A \) is equal to zero, then the matrix is said to be singular. The value of a determinant is determined by obtaining the minors and cofactors of the determinants. The minor of an element \( a_{ij} \) of a determinant of order \( n \) is a determinant of order \( n - 1 \), obtained by removing the row \( i \) and the column \( j \) of the original determinant. The cofactor of a given element of a determinant is the minor of the element with either a plus or minus sign attached as shown below:

\[
\text{cofactor of } a_{ij} = \alpha_{ij} = (-1)^{i+j} M_{ij} \]  \hspace{1cm} (A.20)

where \( M_{ij} \) is the minor of \( a_{ij} \). For example, the cofactor of the element \( a_{23} \) of

\[
\text{det } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\]

\[
\alpha_{23} = (-1)^5 M_{23} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}
\]

The value of a determinant of second order \((2 \times 2)\) is

\[
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} a_{22} - a_{12} a_{21}) \]  \hspace{1cm} (A.21)

The general \( n \)th order determinant has a value given by

\[
\text{det } A = \sum_{j=1}^{n} a_{ij} \alpha_{ij} \text{, with } i \text{ chosen for one row,} \]

or

\[
\text{det } A = \sum_{j=1}^{n} a_{ij} \alpha_{ij} \text{, with } j \text{ chosen for one column.} \]  \hspace{1cm} (A.22)

That is, the elements \( a_{ij} \) are chosen for a specific row (or column) and that the entire row (or column) is expanded according to equation (A.22).

The adjoint matrix of a square matrix \( A \) is formed by replacing each element \( a_{ij} \) by the cofactor \( \alpha_{ij} \) and then transposing the matrix. Therefore

\[
\text{adjoint } A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix} \]  \hspace{1cm} (A.23)
Useful properties of determinants

1. If $A$ and $B$ are both $n \times n$, then $|AB| = |A||B|$.

2. $|A| = |A'|$.

3. If all the elements in any row or in any column are zero, then $|A| = 0$.

4. If any two rows of $A$ are proportional, $|A| = 0$. The same holds for columns.

5. Interchanging any two rows (or any two columns) of a matrix changes the sign of its determinant.

6. Multiplying all elements of any one row (or column) of a matrix $A$ by a scalar $k$ yields a matrix whose determinant is $k|A|$.

7. Any multiple of a row (column) can be added to any other row (column) without changing the value of the determinant.

Problem

Verify the above statements.

Inverse of a matrix

The inverse of a matrix $A$ is defined such that

$$A A^{-1} = A^{-1} A = I \quad (A.24)$$

where $A^{-1}$ is the inverse of $A$.

It is evident that only square matrices (number of rows equal to the number of columns) can possess inverses.

The inverse of a matrix $A$ can be obtained as

$$A^{-1} = \frac{\text{adjoint of } A}{\det A} \quad (A.25)$$

when $\det A$ is not equal to zero.

Product of inverse matrices

Consider the product $C = AB$. Pre-multiplying both sides of the equation by $B^{-1}A^{-1}$ and post-multiplying both sides by $C^{-1}$ results in the relationship

$$B^{-1}A^{-1} = C^{-1} \quad (A.26)$$

Derivative of the inverse matrix

If $A(t)$ is differentiable and possesses an inverse, the derivative of $A^{-1}(t)$ is given by

$$\frac{d}{dt} [A^{-1}(t)] = -A^{-1}(t) \dot{A}(t) A^{-1}(t) \quad (A.27)$$
**Problem**

Derive equation (A.27).

Consider the following vector-matrix equation in partitioned form as indicated by the vertical and horizontal dotted lines.

\[
\begin{bmatrix}
 y_1 \\
 y_2 \\
 y_3 \\
 y_4 \\
 y_5 \\
\end{bmatrix} =
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
 a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
\end{bmatrix}
\]  

(A.28)

Equation (A.28) can be written in terms of submatrices as

\[
\begin{bmatrix}
 y_1 \\
 y_2 \\
\end{bmatrix} =
\begin{bmatrix}
 A_1 & A_2 \\
 A_3 & A_4 \\
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
\end{bmatrix}
\]  

(A.29)

Equation (A.29) can be expanded as

\[
y_1 = A_1 x_1 + A_2 x_2
\]

(A.30)

\[
y_2 = A_3 x_1 + A_4 x_2
\]

(A.31)

From (A.31)

\[
x_2 = A_4^{-1} y_2 - A_4^{-1} A_3 x_1
\]

(A.32)

Substituting (A.32) into (A.30),

\[
y_1 = [A_1 - A_2 A_4^{-1} A_3] x_1 + A_2 A_4^{-1} y_2
\]

(A.33)

Equations (A.33) and (A.32) can be arranged as

\[
\begin{bmatrix}
 y_1 \\
 x_2 \\
\end{bmatrix} =
\begin{bmatrix}
 A_1 - A_2 A_4^{-1} A_3 & A_2 A_4^{-1} \\
 -A_4^{-1} A_3 & A_4^{-1} \\
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 y_2 \\
\end{bmatrix}
\]  

(A.34)

The above procedure demonstrates how a group of variables can be transferred from the right-hand side of equation (A.28) to the left-hand side. If the variables are transferred one at a time and the procedure is repeated \(n\) times for the \(n\) variables, the resulting equation would be, in matrix form,

\[
x = B y
\]

where

\[
B = A^{-1}
\]

The above illustrates an efficient technique for inverting matrices.

In equation (A.29) if \(y_2\) is a null vector (i.e., \(y_4 = y_5 = 0\)) then, from (A.33),

\[
y_1 = [A_1 - A_2 A_4^{-1} A_3] x_1
\]

(A.35)

Equation (A.35) is useful in the elimination of passive nodes in network reduction.
Characteristic values (eigenvalues) and characteristic vectors (eigenvectors)

Consider the vector-matrix equation

\[ y = Ax \]  \hspace{1cm} (A.36)

where \( y \) and \( x \) are column vectors, and \( A \) is a square \( n \times n \) matrix. This equation can be viewed as a transformation of the vector \( x \) into the vector \( y \). The question arises whether there exists a vector \( x \), such that the transformation \( A \) produces a vector \( y \), which has the same direction in vector space as the vector \( x \). If such a vector exists, then \( y \) is proportional to \( x \), or

\[ y = Ax = \lambda x \]  \hspace{1cm} (A.37)

where \( \lambda \) is a scalar of proportionality. This is known as the characteristic value problem, and a value of \( \lambda \), e.g., \( \lambda_i \), for which equation (A.37) has a solution \( (x_i \neq 0) \), is called a characteristic value or eigenvalue of \( A \). The corresponding vector solution \( (x_i \neq 0) \) is called a characteristic vector or eigenvector of \( A \) associated with the characteristic value \( \lambda_i \).

Equation (A.37) can be written in the form

\[ [A - \lambda I]x = 0 \]  \hspace{1cm} (A.38)

This system of homogeneous equations has a nontrivial solution if, and only if, the determinant of the coefficients vanishes, i.e., if

\[ |A - \lambda I| = 0 \]  \hspace{1cm} (A.39)

The \( n \)th order polynomial in \( \lambda \), given by equation (A.39), is called the characteristic equation corresponding to the matrix \( A \). The general form of the equation is

\[ P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n = 0 \]  \hspace{1cm} (A.40)

The roots of the characteristic equation are precisely the characteristic values or eigenvalues of \( A \). When all the eigenvalues of \( A \) are different, \( A \) is said to have distinct roots. When an eigenvalue occurs \( m \) times, the eigenvalue is said to be a repeated root of order \( m \). When a characteristic root is of the form \( \alpha + j\beta \), the root is said to be complex. Complex roots must occur in conjugate pairs, assuming that the elements of \( A \) are real.

Modal matrix

For each of the \( n \) eigenvalues \( \lambda_i \) (\( i = 1, 2, \ldots n \)) of \( A \), a solution of equation (A.38) for \( x \) can be obtained, provided that the roots of equations (A.39) are distinct. The vectors \( x_i \), which are the solutions of

\[ [A - \lambda_i I]x_i = 0 \]  \hspace{1cm} (A.41)

are the characteristic vectors or eigenvectors of \( A \). Since equation (A.41) is homogeneous, \( k_i x_i \), where \( k_i \) is any scalar, is also a solution. Thus only the directions of each of the \( x_i \) are uniquely determined by equation (A.41). The matrix formed by the column vectors \( k_i x_i \) is called the modal matrix.
Diagonalizing a square matrix

Consider the case in which the modal matrix $M$ is non-singular, so that its inverse exists (this is always the case if the eigenvalues of $A$ are distinct). The solution to equation (A.41) can be combined to form the single equation

$$
\begin{pmatrix}
\lambda_1 x_{11} & \lambda_2 x_{12} & \cdots & \lambda_n x_{1n} \\
\lambda_1 x_{21} & \lambda_2 x_{22} & \cdots & \lambda_n x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1 x_{n1} & \lambda_2 x_{n2} & \cdots & \lambda_n x_{nn}
\end{pmatrix}
= 
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}
$$

(A.42)

or

$$
M \Lambda = \Lambda M
$$

(A.43)

where

$$
\Lambda = 
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{pmatrix}
$$

is a diagonal matrix composed of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Since $M^{-1}$ exists, the diagonal matrix $\Lambda$ can be found by pre-multiplying both sides of equation (A.43) by $M^{-1}$, yielding

$$
\Lambda = M^{-1} \Lambda M
$$

(A.44)

Higher powers of $A$ can be diagonalized in the same manner. For example,

$$
\Lambda^2 = [M^{-1} \Lambda M][M^{-1} \Lambda M] = M^{-1} \Lambda^2 M
$$

(A.45)

A transformation of the type $B = Q^{-1}AQ$, where $A$ and $B$ are square matrices and $Q$ is a non-singular square matrix, is called a collineatory or similarity transformation.

**Problem**

Show that the eigenvalues of $B$ are the same as the eigenvalues of $A$.

**Sensitivities of eigenvalues and eigenvectors**

Sensitivities of the eigenvalues and eigenvectors to system parameters or the elements of the system matrix can provide useful information about the system performance.

From the definitions of eigenvalues and eigenvectors we have

$$
Ax = \lambda x
$$

and

$$
A'y = \lambda y
$$

where $y$ is a column vector which is the transpose of the left eigenvector of $A$

$$
y'A = y'\lambda
$$
REVIEW OF MATRICES

\[ \frac{\partial A}{\partial \alpha} x + A \frac{\partial x}{\partial \alpha} = \frac{\partial \lambda}{\partial \alpha} x + \lambda \frac{\partial x}{\partial \alpha} \]

where \( \alpha \) is any parameter of the system. Pre-multiplying both sides by \( y' \)

\[ y' \frac{\partial A}{\partial \alpha} x + y' A \frac{\partial x}{\partial \alpha} = y' \frac{\partial \lambda}{\partial \alpha} x + y' \lambda \frac{\partial x}{\partial \alpha} \]

or

\[ y' \frac{\partial A}{\partial \alpha} x + y' \lambda \frac{\partial x}{\partial \alpha} = y' \frac{\partial \lambda}{\partial \alpha} x + y' \lambda \frac{\partial x}{\partial \alpha} \quad (A.46) \]

From (A.46) it follows that

\[ \frac{\partial \lambda}{\partial \alpha} = \frac{y' \frac{\partial A}{\partial \alpha} x}{y' x} \quad (A.47) \]

When the parameter \( \alpha \) is an element \( a_{ij} \) of the matrix \( A \), all the elements of the matrix \( \frac{\partial A}{\partial \alpha} \) are zero except the one in the \( ij \)th position which is unity. Hence

\[ \frac{\partial \lambda}{\partial a_{ij}} = \frac{y_i x_j}{y' x} \quad (A.48) \]

where \( y_i \) is the \( i \)th element of the vector \( y \) and \( x_j \) is the \( j \)th element of the vector \( x \). The sensitivity matrix \( S_i \), which is the matrix of the sensitivities to each element of \( A \), is thus given by

\[ S_i = \frac{y_i x_i'}{y' x_i} \quad (A.49) \]

for each eigenvalue \( \lambda_i \).

To evaluate \( \frac{\partial x_i}{\partial \alpha} \) we first show that the left and right eigenvectors corresponding to distinct eigenvalues are orthogonal.

We start with

\[ A x_i = \lambda_i x_i \quad (A.50) \]

and

\[ A'y_j = \lambda_j y_j \quad (y' A = y' \lambda_j) \quad (A.51) \]

From (A.50) we can write

\[ y' A x_i = \lambda_i y' x_i \quad (A.52) \]

Similarly, from (A.51)

\[ y' A x_i = \lambda_j y' x_i \quad (A.53) \]

Subtracting (A.53) from (A.52)

\[ 0 = (\lambda_i - \lambda_j) y' x_i \]

Since \( \lambda_i \neq \lambda_j \), it follows that

\[ y' x_i = 0 \quad (A.54) \]

A-11
Now, as before, for the $i$th eigenvalue and eigenvector

$$\frac{\partial A}{\partial \alpha} x_i + A \frac{\partial x_i}{\partial \alpha} = \frac{\partial \lambda_i}{\partial \alpha} x_i + \lambda_i \frac{\partial x_i}{\partial \alpha}$$

Pre-multiplying both sides by $y'_j$

$$y'_j \frac{\partial A}{\partial \alpha} x_i + y'_j A \frac{\partial x_i}{\partial \alpha} = y'_j \frac{\partial \lambda_i}{\partial \alpha} x_i + y'_j \lambda_i \frac{\partial x_i}{\partial \alpha}$$

which yields

$$y'_j \frac{\partial A}{\partial \alpha} x_i = (\lambda_i - \lambda_j) y'_j \frac{\partial x_i}{\partial \alpha} \tag{A.55}$$

Let us set $\frac{\partial x_i}{\partial \alpha} = \sum_{j=1}^{n} \mu_{ij} x_j$, which is a valid assumption as the $n$ linearly independent eigenvectors form a basis for the generation of any $n$ vector, and since eigenvectors are only determined to within a multiplicative constant we are not interested in any perturbation in the direction of $x$, so that we can allow $\mu_{ii}$ to be zero. Thus

$$y'_j \frac{\partial x_i}{\partial \alpha} = y'_j \sum_{j=1}^{n} \mu_{ij} x_j = \mu_{ij} y'_j x_j$$

which yields, from (A.55),

$$\mu_{ij} = \frac{y'_j \frac{\partial A}{\partial \alpha} x_i}{(\lambda_i - \lambda_j) y'_j x_j}$$

and thus $\frac{\partial x_i}{\partial \alpha} = \sum_{j=1}^{n} \mu_{ij} x_j$ is determined.

To obtain the sensitivity of the eigenvector to an element $a_{pq}$ of $A$, we set $\frac{\partial A}{\partial \alpha} = \frac{\partial A}{\partial a_{pq}}$ to obtain

$$\mu_{ij} = \frac{pq\text{th element of } y_j x'_i}{(\lambda_i - \lambda_j) y'_j x_j}$$

**References**

The Laplace Transform

The Laplace transformation is a functional transformation in that it changes the expression for a given quantity from a function of time, \( f(t) \), to a function of the complex operator, \( F(s) \), where \( s \) is the complex operator.

A real function \( f(t) \) has a Laplace transform if it is defined and single-valued almost everywhere for \( 0 \leq t \), and if \( f(t) \) is such that

\[
\int_{0}^{\infty} f(t) e^{-\sigma t} dt < \infty
\]

for some real number \( \sigma \). Then \( f(t) \) is said to be \( L \)-transformable. The functions of time that describe the performance of actual control systems are in this category.

The Laplace transformation for a function of time, \( f(t) \), is defined as

\[
F(s) = \int_{0}^{\infty} f(t) e^{-st} dt = L[f(t)]
\]

where \( s \) is a complex variable such that its real part is greater than \( \sigma \), the number described in the definition for \( L \)-transformability.

The inverse transformation is generally defined implicitly by the definition

\[
f(t) = L^{-1}[F(s)], \quad 0 \leq t
\]

Some important Laplace transform pairs are given in Table B.1.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit step</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{1}{s+a} )</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-at} f(t) )</td>
<td>( F(s+a) )</td>
</tr>
</tbody>
</table>
In the solution of differential equations by the Laplace transform method, the differential equations are transformed into algebraic equations and the resulting algebraic equations are solved for the transform of the variable of interest. The time response solution is then obtained by an inverse Laplace transformation.

In the context of differential equations, the Laplace variable $s$ can be considered to be the differential operator, so that

\[ s = \frac{d}{dt}. \]  \hspace{1cm} (B.4)

Then, we also have the integral operator

\[ \frac{1}{s} = \int_{0}^{t} dt \]  \hspace{1cm} (B.5)

In order to illustrate the use of Laplace transformation consider the spring-mass-damper system described in Chapter 2, which is

\[ M \frac{d^2 y}{dt^2} + f \frac{dy}{dt} + Ky = u(t) \]  \hspace{1cm} (B.6)

Taking the Laplace transform of (B.6),

\[ M \left[ s^2 Y(s) - s y(0^+) - \frac{dy}{dt}(0^+) \right] + f \left[ s Y(s) - y(0^+) \right] + K Y(s) = U(s) \]  \hspace{1cm} (B.7)

When

\[ u(t) = 0, \quad y(0^+) = y_o, \quad \frac{dy}{dt} \bigg|_{t=0} = 0 \]

we have

\[ M s^2 Y(s) - M s y_o + f s Y(s) - f y_o + K Y(s) = 0 \]  \hspace{1cm} (B.8)

Solving for $Y(s)$, we obtain
The denominator polynomial $q(s)$ is the characteristic equation, and the roots of this equation determine the character of the time response. These roots are also called the poles or singularities of the system. The roots of the numerator polynomial $p(s)$ are called the zeros of the system. At the poles the function $Y(s)$ becomes infinite; while at the zeros, the function becomes zero. The complex $s$-plane plot of the poles and zeros graphically portrays the character of the natural transient response of the system.

The time response solution $y(t)$ is obtained by taking the inverse Laplace transform of equation (B.9).

The steady-state or final value of the response $y(t)$ is obtained from the relation

$$
\lim_{t \to \infty} y(t) = \lim_{s \to 0} s Y(s)
$$

where a simple pole of $Y(s)$ at the origin is permitted, but poles on the imaginary axis and in the right-half plane and higher-order poles at the origin are excluded.

The initial value of the response $y(t)$ is obtained from

$$
\lim_{t \to 0} y(t) = \lim_{s \to \infty} s Y(s)
$$

The transfer function of linear systems

The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. The transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration.

A transfer function may only be defined for a linear, stationary (constant parameter) system. Also, the transfer function description does not include any information concerning the internal structure of the system and its behavior.

The transfer function of the spring-mass-damper system is obtained from the original describing equation (B.7), rewritten with zero initial conditions as follows:

$$
M s^2 Y(s) + f s Y(s) + K Y(s) = U(s)
$$

Then the transfer function is

$$
G(s) = \frac{Y(s)}{U(s)} = \frac{1}{M s^2 + f s + K}
$$

The transfer function of the RC network shown in Figure B.1 is obtained by writing the voltage equations, in transformed form, yielding

$$
V_1(s) = \left( R + \frac{1}{Cs} \right) I(s)
$$

$$
V_2(s) = \frac{1}{Cs} I(s)
$$
From (B.14) and (B.15) we obtain the transfer function as

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1} = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}$$  \hspace{1cm} (B.16)

where \( \tau = RC \), the time constant of the network.

**Block diagrams**

The dynamic systems that comprise automatic control systems are represented mathematically by a set of simultaneous differential equations. The introduction of the Laplace transformation reduces the problem to the solution of a set of linear algebraic equations. Since control systems are concerned with the control of specific variables, the interrelationship of the controlled variables to the controlling variables is required. This relationship is typically represented by the transfer functions of the subsystems relating the input and output variables in the form of block diagrams. Block diagrams consist of unidirectional, operational blocks that represent the transfer function of the variables of interest. A block diagram of the spring-mass-damper system is shown in Figure B.2.

In order to represent a system with several variables under control, an interconnection of blocks is utilized.

The block diagram representation of a given system may often be reduced by block diagram reduction techniques to a simplified block diagram with fewer blocks than in the original diagram.

Block diagram transformation and reduction techniques are derived by considering the algebra of the diagram variables. For example, consider the block diagram shown in Figure B.3.
The negative feedback control system is described by the equation for the actuating signal

\[ E_a(s) = U(s) - B(s) = U(s) - H(s)Y(s) \]  

(B.17)

Also, we have

\[ Y(s) = G(s)E_a(s) \]  

(B.18)

Therefore

\[ Y(s) = G(s)[U(s) - H(s)Y(s)] \]

Solving for \( Y(s) \) we obtain

\[ Y(s)[1 + G(s)H(s)] = G(s)U(s) \]

Therefore, the transfer function relating the output \( Y(s) \) to the input \( U(s) \) is

\[ \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} \]  

(B.19)

This closed-loop transfer function is particularly important since it represents many of the existing practical control systems. The reduced block diagram is shown in Figure B.4

The above is one example of several useful block diagram reductions. These diagram transformations are given in Table B.2. All the transformations in Table B.2 can be derived by simple algebraic manipulation of the equations representing the blocks. System analysis by the method of block diagram reduction has the advantage of affording a better understanding of the contribution of each component element than is possible by the manipulation of equations.
A control system can be defined as an interconnection of components forming a system configuration which will provide a desired system response. Since a desired system response is known, a signal proportional to the error between the desired and the actual response is generated. The utilization of this signal to control the process results in a closed-loop sequence of operations which is called a feedback system.

### Open- and closed-loop control systems

A control system can be defined as an interconnection of components forming a system configuration which will provide a desired system response. Since a desired system response is known, a signal proportional to the error between the desired and the actual response is generated. The utilization of this signal to control the process results in a closed-loop sequence of operations which is called a feedback system.

---

**Table B.2  Block diagram transformation**

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Original Diagram</th>
<th>Equivalent Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Combining blocks in cascade</td>
<td><img src="image1" alt="Original Diagram" /></td>
<td><img src="image2" alt="Equivalent Diagram" /></td>
</tr>
<tr>
<td>2. Moving a summing point behind a block</td>
<td><img src="image3" alt="Original Diagram" /></td>
<td><img src="image4" alt="Equivalent Diagram" /></td>
</tr>
<tr>
<td>3. Moving a pickoff point ahead of a block</td>
<td><img src="image5" alt="Original Diagram" /></td>
<td><img src="image6" alt="Equivalent Diagram" /></td>
</tr>
<tr>
<td>4. Moving a pickoff point behind a block</td>
<td><img src="image7" alt="Original Diagram" /></td>
<td><img src="image8" alt="Equivalent Diagram" /></td>
</tr>
<tr>
<td>5. Moving a summing point ahead of a block</td>
<td><img src="image9" alt="Original Diagram" /></td>
<td><img src="image10" alt="Equivalent Diagram" /></td>
</tr>
<tr>
<td>6. Eliminating a feedback loop</td>
<td><img src="image11" alt="Original Diagram" /></td>
<td><img src="image12" alt="Equivalent Diagram" /></td>
</tr>
</tbody>
</table>
In order to illustrate the characteristics and advantages of introducing feedback, we shall consider a simple, single-loop feedback system. An open-loop control system is shown in Figure B.5a, and a closed-loop, negative feedback control system is shown in Figure B.5b.

The main difference between the open- and closed-loop systems is the generation and utilization of the error signal. The closed-loop system, when operating correctly, operates so that the error will be reduced to a minimum value. The signal $E_a(s)$ is a measure of the error of the system and is equal to the error $E(s) = U(s) - Y(s)$ when $H(s) = 1$.

The output of the open-loop system is

$$Y(s) = G(s)U(s) \quad (B.20)$$

The output of the closed-loop system is

$$Y(s) = G(s)E_a(s) = G(s)[U(s) - H(s)Y(s)]$$

from which

$$Y(s) = \frac{G(s)}{1 + GH(s)}U(s) \quad (B.21)$$

The actuating error signal is

$$E_a(s) = \frac{1}{1 + GH(s)}U(s) \quad (B.22)$$

Therefore, in order to reduce the error, the magnitude of $1 + GH(s)$ must be greater than one over the range of $s$ under consideration.

**Sensitivity of control systems to parameter variations**

A process $G(s)$, whatever its nature, is subject to change. In the open-loop system this change will result in a changing and inaccurate output. A closed-loop system senses the change in the output due to the process changes and attempts to correct the output. A primary advantage of a closed-loop feedback control system is its ability to reduce the system's sensitivity.

For the closed-loop case, if $GH(s) \gg 1$, then from equation (B.21),

$$Y(s) \approx \frac{1}{H(s)}U(s) \quad (B.23)$$
That is, the output is only affected by \( H(s) \) which may be a constant. If \( H(s) = 1 \), the output is equal to the input. However, it should be noted that \( GH(s) \gg 1 \) may cause the system response to be highly oscillatory and even unstable. But the fact that as the magnitude of the loop transfer function \( GH(s) \) is increased the effect of \( G(s) \) on the output is reduced, is a useful concept.

In order to illustrate the effect of parameter variations, let us look at the incremental equations derived from (B.20) and (B.21).

From (B.20), the change in the transform of the output is
\[
\Delta Y(s) = \Delta G(s)U(s) \tag{B.24}
\]
From (B.21), the change in the output is
\[
\Delta Y(s) = \frac{\Delta G(s)}{[1 + GH(s)]^2} U(s) \tag{B.25}
\]
Comparing equations (B.24) and (B.25), we note that the change in the output of the closed-loop system is reduced by a factor \([1 + GH(s)]^2\) which is usually much greater than one.

If system sensitivity is defined as the ratio of the percentage change in the system transfer function to the percentage change in the process transfer function,
\[
S = \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)} \tag{B.26}
\]
In the limit (B.26) reduces to
\[
S = \frac{\partial T / T}{\partial G / G} = \frac{\partial \ln T}{\partial \ln G} \tag{B.27}
\]
From (B.20) it can be seen that the sensitivity of the open loop system is equal to one.

The sensitivity of the closed-loop system is obtained from equation (B.21). The system transfer function is
\[
T(s) = \frac{G(s)}{1 + GH(s)}
\]
Therefore, the sensitivity of the feedback system is
\[
S = \frac{\partial T G}{\partial G T} = \frac{1}{1 + GH(s)} \tag{B.28}
\]
Thus the sensitivity of the system may be reduced below that of the open-loop system by increasing \( GH(s) \).

**Control of the transient response of control systems**

One of the most important characteristics of control systems is their transient response. Since the purpose of control systems is to provide a desired response, the transient response of control systems often must be adjusted until it is satisfactory. If an open-loop control system does not provide a satisfactory response, then the process, \( G(s) \), must be replaced with a suitable process. By contrast, a closed-loop system can often be adjusted to yield the desired response by adjusting the feedback loop parameters.
As an example, consider the open-loop system with a transfer function as shown in Figure B.6.

![Open-loop control system diagram](image)

**Fig. B.6** An open-loop control system.

For a unit step input, the output response is

\[ Y(s) = G(s) \frac{1}{s} = \frac{K}{\tau s + 1} \]

The transient time response is then

\[ y(t) = K \left( 1 - e^{-t/\tau} \right) \]  

Now consider the closed-loop system shown in Figure B.7, where the error signal is amplified before being fed to the process. The amplifier gain \( K_a \) may be adjusted to meet the required transient response specifications. The feedback gain, \( K_f \), may also be varied.

![Closed-loop control system diagram](image)

**Fig. B.7** A closed-loop control system.

The closed-loop transfer function is

\[ \frac{Y(s)}{U(s)} = \frac{\frac{K}{\tau s + 1}}{1 + \frac{K_a \cdot K}{\tau s + 1}} = \frac{K_a K}{\tau s + 1 + K_f K} \]  

Therefore, the transient response for a unit step input is

\[ y(t) = \frac{K_a K}{1 + K_a K_f} \left( 1 - e^{-\left(1 + K_a K_f\right)t/\tau} \right) \]

For a large value of the amplifier gain, \( K_a \), and for \( K_f K = 1 \), the approximate response is

\[ y(t) = K \left( 1 - e^{-K_f t/\tau} \right) \]

Comparing (B.33) with (B.30), it can be seen that using a closed-loop system considerable improvement in the speed of response can be achieved.
Stability of linear feedback systems

For linear systems the stability requirement may be defined in terms of the location of the poles of the closed-loop transfer function. The closed-loop system transfer function is written as

$$T(s) = \frac{p(s)}{q(s)}$$ (B.34)

where $q(s)$ is the characteristic equation whose roots are the poles of the closed-loop system. A necessary and sufficient condition that a feedback system be stable is that all the poles of the system transfer function have negative real parts, i.e., the poles are in the left-hand side of the s-plane.

In order to ascertain the stability of a feedback control system, one could determine the roots of the characteristic equation $q(s)$. However, if we are interested only in the knowledge of stability or instability of a system, several simpler methods are available.

The Routh-Hurwitz stability criterion

The Routh-Hurwitz stability method provides an answer to the question of stability by considering the characteristic equation of the system. The characteristic equation in Laplace variable is written as

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$ (B.35)

In order to ascertain the stability of the system, it is necessary to determine if any of the roots of $q(s)$ lie in the right-half of the s-plane.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems. The criterion is based on ordering the coefficients of the characteristic equation (B.35) into an array as follows:

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
s^0 & h_{n-1} & & & \cdots \\
\end{array}$$

where

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-3}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

and so on.
The Routh-Hurwitz criterion states that the number of roots of \( q(s) \) with positive real parts is equal to the number of changes in sign of the first column of the array. This criterion requires that there be no changes in sign in the first column for a stable system. This requirement is both necessary and sufficient.

**The root locus method**

The relative stability and the transient performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation in the \( s \)-plane. Also, it is frequently necessary to adjust one or more system parameters in order to obtain suitable root locations. The root locus technique is a graphical method for drawing the locus of roots in the \( s \)-plane as a parameter is varied.

The dynamic performance of a closed-loop control system is described by the closed-loop transfer function

\[
T(s) = \frac{Y(s)}{U(s)} = \frac{p(s)}{q(s)}
\]  

(B.36)

where \( p(s) \) and \( q(s) \) are polynomials in \( s \). The roots of the characteristic equation \( q(s) \) determine the modes of response of the system. For a closed-loop system the characteristic equation may be written as

\[
q(s) = 1 + F(s)
\]  

(B.37)

The roots of the characteristic equation is obtained from

\[
1 + F(s) = 0
\]  

(B.38)

From (B.38) it follows that the roots of the characteristic equation must satisfy the relation

\[
F(s) = -1
\]  

(B.39)

In the case of the simple single-loop system shown in Figure B.5b, the characteristic equation is

\[
1 + G H(s) = 0
\]  

(B.40)

where

\[
F(s) = G H(s)
\]

The characteristic roots must satisfy equation (B.39). Since \( s \) is a complex variable, (B.39) may be written in polar form as

\[
|F(s)| \angle F(s) = -1
\]  

(B.41)

and therefore it is necessary that

\[
|F(s)| = 1
\]

and

\[
\angle F(s) = 180^\circ \pm k 360^\circ
\]  

(B.42)

\[k = 0, 1, 2, \cdots\]

As an example consider the system shown in Figure B.8
The characteristic equation representing this system is

\[ 1 + G(s) = 1 + \frac{K}{s(s + a)} = 0 \]

or alternatively

\[ q(s) = s^2 + a s + K = s^2 + 2 \zeta \omega_n s + \omega_n^2 = 0 \]  \hspace{1cm} (B.43)

The locus of the roots as the gain \( K \) is varied is found by requiring that

\[ |G(s)| = \left| \frac{K}{s(s + a)} \right| = 1 \]  \hspace{1cm} (B.44)

and

\[ \angle G(s) = \pm 180^\circ, \pm 540^\circ, \cdots \]  \hspace{1cm} (B.45)

In this simple example, we know that the roots are

\[ s_1, s_2 = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]  \hspace{1cm} (B.46)

As the gain \( K \) is varied from zero to a very large positive value, it can be seen that the roots follow a locus as shown in Figure B.9, starting at the location of the open-loop poles \((0, -a)\) for \( K = 0 \). For \( \zeta < 1 \) the locus of the roots is a vertical line in order to satisfy the angle requirement, equation (B.45). The angle requirement is satisfied at any point on the vertical line which is a perpendicular bisector of the line \(0 \) to \( a\). Furthermore, the gain \( K \) at the particular point \( s_1 \) is found from equation (B.44) as

\[ K = \frac{1}{|s_1| |s_1 + a|} \]

or

\[ K = |s_1| |s_1 + a| \]  \hspace{1cm} (B.47)

where \( |s_1| \) is the magnitude of the vector from the origin to \( s_1 \), and \( |s_1 + a| \) is the magnitude of the vector from \(-a\) to \( s_1\).
In order to locate the roots of the characteristic equation in a graphical manner on the $s$-plane, an orderly procedure which facilitates the rapid sketching of the locus is required. The characteristic equation is first written as
\[ 1 + F(s) = 0 \]

The equation is rearranged, if necessary, so that the parameter of interest, $K$, appears as the multiplying factor in the form
\[ 1 + K p(s) = 0 \quad \text{(B.48)} \]

$p(s)$ is factored and the polynomial is written in the form of poles and zeros as follows:
\[ 1 + K \prod_{j=1}^{M} (s + z_j) = 0 \quad \text{(B.49)} \]

The poles and zeros are located on the $s$-plane with appropriate markings. One is usually interested in determining the locus of roots as $K$ varies from 0 to $\infty$.

Rewriting equation (B.49), we have
\[ \prod_{j=1}^{N} (s + p_j) + K \prod_{j=1}^{M} (s + z_j) = 0 \quad \text{(B.50)} \]

Therefore, when $K = 0$, the roots of the characteristic equation are simply the poles of $p(s)$. Also, when $K \to \infty$, the roots of the characteristic equation are simply the zeros of $p(s)$.

Therefore, we note that the locus of the roots of the characteristic equation $1 + K p(s) = 0$ begins at the poles of $p(s)$ and ends at the zeros of $p(s)$ as $K$ increases from 0 to $\infty$. In most cases several of the zeros of $p(s)$ lie at infinity in the $s$-plane.

The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros. This fact is ascertained by examining the angle criterion for the root locus.
Frequency response methods

A very practical alternative approach to the analysis and design of a system is the frequency response method. The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. In a linear system, when the input signal is sinusoidal, the resulting output signal as well as signals throughout the system, is sinusoidal in the steady state; it differs from the input waveform only in amplitude and phase angle.

An advantage of the frequency response method is that the transfer function describing the sinusoidal steady-state behavior of a system can be obtained by replacing $s$ with $j\omega$ in the system transfer function $T(s)$. The transfer function representing the sinusoidal steady-state behavior of a system is then a function of the complex variable $j\omega$ and is itself a complex function $T(j\omega)$ which possesses a magnitude and phase angle. The magnitude and phase angle of $T(j\omega)$ are readily represented by graphical plots which provide a significant insight for the analysis and design of control systems.

Direct correlations between the frequency response and the corresponding transient response characteristics are somewhat tenuous, and in practice the frequency response characteristic is adjusted by using various design criteria which will normally result in a satisfactory transient response.

Frequency response plots

The transfer function of a system $G(s)$ can be described in the frequency domain by the relation

$$G(j\omega) = G(s)|_{s=j\omega} = R(j\omega) + jX(j\omega)$$  \hspace{1cm} (B.51)$$

where

$$R(j\omega) = \text{Re}[G(j\omega)], \quad X(j\omega) = \text{Im}[G(j\omega)]$$

Alternatively, the transfer function can be represented by a magnitude $|G(j\omega)|$ and a phase $\phi(j\omega)$ as

$$G(j\omega) = |G(j\omega)|e^{j\phi(j\omega)} = |G(j\omega)| \angle \phi(j\omega)$$  \hspace{1cm} (B.52)$$

where

$$|G(j\omega)|^2 = [R(j\omega)]^2 + [X(j\omega)]^2$$

and

$$\phi(j\omega) = \tan^{-1} \frac{X(j\omega)}{R(j\omega)}$$

In order to simplify the determination of the graphical portrayal of the frequency response, logarithmic plots, often called Bode plots, are employed.

The natural logarithm of equation (B.52) is

$$\ln G(j\omega) = \ln |G(j\omega)| + j\phi(j\omega)$$  \hspace{1cm} (B.53)$$

where $\ln |G(j\omega)|$ is the magnitude in nepers. The logarithm of the magnitude is normally expressed in terms of the logarithm to the base 10, and the following definition is employed.

$$\text{Logarithmic gain} = 20 \log_{10} |G(j\omega)|$$
where the units are decibels (db). For a bode diagram, the plot of logarithmic gain in db vs $\omega$ is normally plotted on one set of axes and the phase $\phi(j\omega)$ vs $\omega$ on another set of axes.

As an example, consider the simple $RC$ filter shown in Figure B.10.

![An RC filter.](image_url)

The transfer function of this filter is

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1}$$

and the sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega\tau + 1}$$  \hspace{1cm} (B.54)

where $\tau = RC$, the time constant of the network.

The logarithmic gain is

$$20\log|G| = 20\log\left(\frac{1}{1 + (\omega\tau)^2}\right)^{\frac{1}{2}} = -10\log[1 + (\omega\tau)^2]$$  \hspace{1cm} (B.55)

For small frequencies, i.e., $\omega << 1/\tau$, the logarithmic gain is

$$20\log|G| = -10\log1 = 0\text{db}$$  \hspace{1cm} (B.56)

For large frequencies, i.e., $\omega >> 1/\tau$, the logarithmic gain is

$$20\log|G| = -20\log\omega\tau$$  \hspace{1cm} (B.57)

At $\omega = 1/\tau$, we have

$$20\log|G| = -10\log2 = -3.01\text{db}$$  \hspace{1cm} (B.58)

The magnitude plot for this network is shown in Figure B.11a.

The phase angle of this network is

$$\phi(j\omega) = -\tan^{-1}\omega\tau$$  \hspace{1cm} (B.59)

The phase plot is shown in Figure B.11b. The frequency $\omega = 1/\tau$ is often called the break frequency or corner frequency.
A linear scale of frequency is not the most convenient choice in plotting the Bode diagram. A logarithmic scale of frequency is generally employed. The convenience of a logarithmic scale of frequency can be seen by considering equation (B.57) for large frequencies, \( \omega >> 1/\tau \), as follows:

\[
20 \log |G| = -20 \log \omega \tau = -20 \log \tau - 20 \log \omega \tag{B.60}
\]

Then, on a set of axes where the horizontal axis is \( \log \omega \), the asymptotic curve for \( \omega >> 1/\tau \) is a straight line as shown in Figure B.12.

A ratio of two frequencies equal to ten is called a decade, so that the difference between the logarithmic gains, for \( \omega >> 1/\tau \), over a decade of frequency is

\[
20 \log |G(\omega_1)| - 20 \log |G(\omega_2)| = -20 \log \omega_1 \tau - (-20 \log \omega_2 \tau) = -20 \log \frac{\omega_1 \tau}{\omega_2 \tau} = -20 \log \frac{10}{1} = +20 \text{ db} \tag{B.61}
\]

That is, the slope of the asymptotic line for this first-order transfer function is \(-20 \text{ db/decade}\), as shown in Figure B.12.

References
APPENDIX C

BLOCK DIAGRAM STATE MODEL

The dynamics of a system, in state-space form, are represented by a set of first-order differential equations in terms of a set of state variables, which in compact form may be represented by a vector differential equation of the form

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (C.1)

as explained in Chapter 2.

It is useful to develop a block diagram state model of the system and use this model to relate the state variable concept to the transfer function representation.

We have seen that a system can be described by an input-output relationship -- the transfer function $G(s)$. For example, if we are interested in the relationship between the output voltage and the input voltage of the network shown in Figure C.1, we may obtain the transfer function

The differential equations describing the circuit, choosing state variables $x_1 = V_C$ and $x_2 = i$, are

$$\frac{dx_1}{dt} = \frac{1}{C} x_2$$ \hspace{1cm} (C.3)

$$\frac{dx_2}{dt} = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u(t)$$ \hspace{1cm} (C.4)

The output variable is

$$y(t) = V_C = x_1$$ \hspace{1cm} (C.5)

The block diagram representing these simultaneous equations is shown in Figure C.2, where $1/s$ indicates an integration.
Using the block diagram reduction technique we obtain the transfer function as
\[
\frac{Y(s)}{U(s)} = \frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} \quad \text{(C.6)}
\]

A block diagram state model of a system is useful in that the system differential equations can be written down directly from the diagram. The diagram may also be recognized as being equivalent to an analog computer diagram. Also, since there can be more than one set of state variables describing a system, there can be more than one possible form for the block diagram state model.

**Problem**

Draw the block diagram state model for the system described by the following set of equations.

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \\
y &= c_{11}x_1 + c_{12}x_2
\end{align*}
\]

It is often a difficult task to determine a set of first-order differential equations describing the system. This may be, for example, due to a lack of information concerning the internal structure of the system and its behavior. Frequently it is simpler to derive the transfer function of a system or deduce it from the experimentally determined frequency response.

The block diagram state model can be readily derived from the transfer function of a system. In general, a transfer function may be represented as
\[
G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad \text{(C.7)}
\]

where \(n \geq m\) and all the \(a\) and \(b\) coefficients are real positive numbers.

In order to illustrate the derivation of the block diagram state model, let us first consider the fourth-order transfer function
\[
G(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \quad \text{(C.8)}
\]
We note that the system is of fourth-order, and hence we will require four state variables. Choosing the state variables as

\[\begin{align*}
  x_1 &= y \\
  x_2 &= \dot{y} = sy \\
  x_3 &= \dot{\dot{y}} = s^2y \\
  x_4 &= \ddot{\dot{y}} = s^3y
\end{align*}\]

We can write the differential equations corresponding to the transfer function (C.8) as

\[\begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= x_3 \\
  \dot{x}_3 &= x_4 \\
  \dot{x}_4 &= (u - a_3x_4 - a_2x_3 - a_1x_2 - a_0x_1)/a_4
\end{align*}\]

(C.9)

The block diagram state model corresponding to the above differential equations or the transfer function of equation (C.8) may therefore be drawn as shown in Figure C.3

![Block diagram state model](image)

**Fig. C.3** A block diagram state model for the transfer function of equation C.8.

Next consider the fourth-order transfer function when the numerator is also a polynomial in \(s\) so that

\[G(s) = \frac{Y(s)}{U(s)} = \frac{b_4s^4 + b_3s^3 + b_2s^2 + b_1s + b_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}\]

(C.10)

Defining a variable \(C(s)\) as

\[C(s) = \frac{Y(s)}{b_4s^4 + b_3s^3 + b_2s^2 + b_1s + b_0}\]

(C.11)

equation (C.10) can be reduced to

\[\frac{C(s)}{U(s)} = \frac{1}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}\]

(C.12)

which is of the same form as (C.8).
As before, defining state variables

\[
\begin{align*}
    x_1 &= c \\
    x_2 &= \dot{c} = sc \\
    x_3 &= \ddot{c} = s^2 c \\
    x_4 &= \dddot{c} = s^3 c
\end{align*}
\]

the differential equations corresponding to equation (C.12) will be as in equation (C.9) and the block diagram state model will be as in Figure C.3.

From (C.11)

\[
Y(s) = C(s)[b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0]
\]

Therefore, a block diagram state model for the transfer function of equation (C.10) may be drawn as shown in Figure C.4.

![Block Diagram State Model](image)

**Fig. C.4** A block diagram state model for the transfer function of equation C.10.

A block diagram state model for any of the transfer functions describing the dynamics of the excitation or governor control systems may be readily obtained from Figures C.3 and C.4.

**Examples**

A block diagram model for the transfer function \(G(s) = 1/(\tau s + 1)\) may be obtained by comparing it with the transfer function of equation (C.8). The block diagram follows from Figure C.3 and is shown in Figure C.5.
A block diagram state model for the transfer function \( G(s) = (\tau_2 s + 1) / (\tau_1 s + 1) \) may be obtained by comparing it with the transfer function of equation (C.10). The block diagram follows from Figure C.4 and is shown in Figure C.6.

In order to illustrate how easily the set of first-order differential equations describing a system may be obtained once the transfer function block diagram model has been converted into a state model, we will consider the excitation control model shown in Figure C.7.

The state model for the block diagram of Figure C.7 is shown in Figure C.8.
The set of first-order differential equations for the system is

\[
\begin{align*}
\dot{x}_1 &= [V_{\text{ref}} - V_1 - K_F (E_{fd} - x_2)] / \tau_F - x_1 ] / \tau_A \\
\dot{x}_2 &= (E_{fd} - x_2) / \tau_F 
\end{align*}
\]

References